GOVERNMENT POLYTECHNIC MANESAR
E-CONTENTS STRENGTH OF MATERIAL

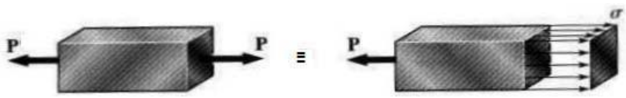
Chapter – 1 Stress and Strain

1.1 Stress (σ)

When a material is subjected to an external force, a resisting force is set up within the component. The internal resistance force per unit area acting on a material or intensity of the forces distributed over a given section is called the stress at a point.

- It uses original cross section area of the specimen and also known as engineering stress or conventional stress.

  Therefore, $\sigma = \dfrac{P}{A}$



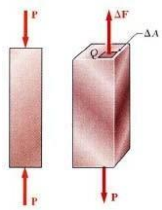
- *P is* expressed in *Newton* (N) and *A*, original area, in square meters (m^2), the stress σ will be expresses in N/ m^2. This unit is called *Pascal (Pa).*
- As *Pascal* is a small quantity, in practice, multiples of this unit is used.

1 kPa = 10^3 Pa = 10^3 N/ m^2	(kPa = Kilo Pascal)
1 MPa = 10^6 Pa = 10^6 N/ m^2 = 1 N/mm^2	(MPa = Mega Pascal)
1 GPa = 10^9 Pa = 10^9 N/ m^2	(GPa = Giga Pascal)

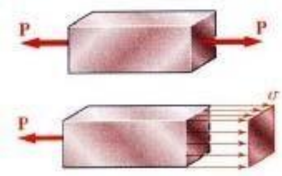
Let us take an example: A rod 10 mm × 10 mm cross-section is carrying an axial tensile load 10 kN. In this rod the tensile stress developed is given by

$$(\sigma_t) = \frac{P}{A} = \frac{10kN}{(10\,mm \times 10mm)} = \frac{10 \times 10^3 N}{100\,mm^2} = 100 N/mm^2 = 100 MPa$$

- The resultant of the internal forces for an axially loaded member is normal to a section cut perpendicular to the member axis.
- The force intensity on the shown section is defined as the normal stress. $\sigma = \lim_{\Delta A \to 0} \dfrac{\Delta F}{\Delta A}$ and $\sigma_{avg} = \dfrac{P}{A}$

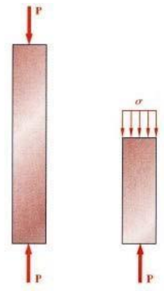


- **Tensile stress (σ_t)**



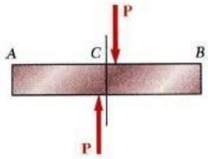
If $\sigma > 0$ the stress is tensile. i.e. The fibres of the component tend to elongate due to the external force. A member subjected to an external force tensile P and tensile stress distribution due to the force is shown in the given figure.

- **Compressive stress (σ_c)**

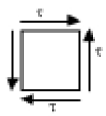


If $\sigma < 0$ the stress is compressive. i.e. The fibres of the component tend to shorten due to the external force. A member subjected to an external compressive force P and compressive stress distribution due to the force is shown in the given figure.

- **Shear stress (τ)**



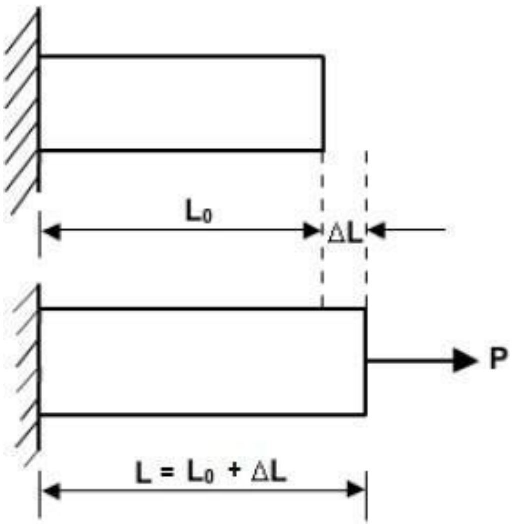
When forces are transmitted from one part of a body to other, the stresses developed in a plane parallel to the applied force are the shear stress. Shear stress acts parallel to plane of interest. Forces *P* is applied transversely to the member *AB as shown. The c*orresponding internal forces act in the plane of section *C* and are called *shearing*



forces. The corresponding average shear stress (τ)= $\dfrac{P}{Area}$

1.2 Strain (ε)

The displacement per unit length ***(dimensionless)*** is known as strain.



- **Tensile strain (εt)**

The elongation per unit length as shown in the figure is known as tensile strain.

$$\varepsilon_t = \Delta L / L_o$$

It is engineering strain or conventional strain. Here we divide the elongation to original length not actual length ($L_o + \Delta L$)

Let us take an example: A rod 100 mm in original length. When we apply an axial tensile load 10 kN the final length of the rod after application of the load is 100.1 mm. So in this rod tensile strain is developed and is given by

$$(\varepsilon) = \frac{\Delta L}{L_o} = \frac{L - L_o}{L_o} = \frac{100.1\,mm - 100\,mm}{100\,mm} = \frac{0.1\,mm}{100\,mm} = 0.001 \text{ (Dimensionless) Tensile}$$

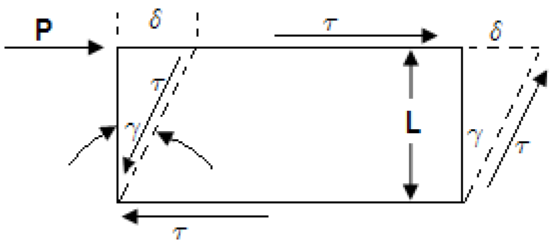
- **Compressive strain (ε_c)**

  If the applied force is compressive then the reduction of length per unit length is known as compressive strain. It is negative. Then $\varepsilon_c = (-\Delta L)/ L_o$

Let us take an example: A rod 100 mm in original length. When we apply an axial compressive load 10 kN the final length of the rod after application of the load is 99 mm. So in this rod a compressive strain is developed and is given by

$$(\varepsilon_c) = \frac{\Delta L}{L_o} = \frac{L - L_o}{L_o} = \frac{99\,mm - 100\,mm}{100\,mm} = \frac{-1\,mm}{100\,mm} = -0.01 \text{ (Dimensionless) compressive}$$

- **Shear Strain (γ):** When a force P is applied tangentially to the element shown. Its edge displaced to dotted line. Where δ is the lateral displacement of the upper face of the element relative to the lower face and L is the distance between these faces.



Then the shear strain is $(\gamma) = \frac{\delta}{L}$

Let us take an example: A block 100 mm × 100 mm base and 10 mm height. When we apply a tangential force 10 kN to the upper edge it is displaced 1 mm relative to lower face.
Then the direct shear stress in the element

$$(\tau) = \frac{10\,kN}{100\,mm \times 100\,mm} = \frac{10 \times 10^3\,N}{100\,mm \times 100\,mm} = 1\text{ N/mm}^2 = 1 \text{ MPa}$$

And shear strain in the element (γ) = $= \frac{1\,mm}{10\,mm} = 0.1$ Dimensionless

1.3 True stress and True Strain

The true stress is defined as the ratio of the load to the cross section area at any instant.

$$(\sigma_T) = \frac{\text{load}}{\text{Instantaneous area}} = \sigma(1+\varepsilon)$$

Where σ and ε is the engineering stress and engineering strain respectively.

- **True strain**

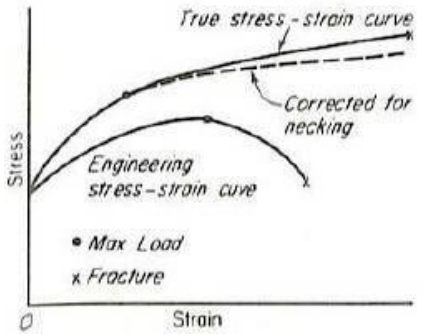
$$(\varepsilon_T) = \int_{L_o}^{L} \frac{dl}{l} = \ln\left(\frac{L}{L_o}\right) = \ln(1+\varepsilon) = \ln\left(\frac{A_o}{A}\right) = 2\ln\left(\frac{d_o}{d}\right)$$

or engineering strain (ε) = e^{ε_T} -1

The volume of the specimen is assumed to be constant during plastic deformation.

[∵ $A_O L_O = AL$] It is valid till the neckformation.

- **Comparison of engineering and the true stress-strain curves shown below**
- The true stress-strain curve is also known as the **flow curve.**
- True stress-strain curve gives a true indication of deformation characteristics because it is based on the instantaneous dimension of the specimen.
- In engineering stress-strain curve, stress drops down after necking since it is based on the original area.



- In true stress-strain curve, the stress however increases after necking since the cross-sectional area of the specimen decreases rapidly after necking.
- The flow curve of many metals in the region of uniform plastic deformation can be expressed by the **simple power law.**

$\sigma_T = K(\varepsilon_T)^n$ Where K is the strength coefficient

n is the strain hardening exponent

n = 0 perfectly plastic solid

n = 1 elastic solid

For most metals, 0.1< n < 0

1.4 Hook's law

According to Hook's law the stress is directly proportional to strain i.e. normal stress (σ) α normal strain (ε) and shearing stress (τ) α shearing strain (γ).

$$\sigma = E\varepsilon \text{ and } \tau = G\gamma$$

The co-efficient E is called the *modulus of elasticity i. e i.e. its resistance to elastic strain.* The co-efficient G is called the *shear modulus of elasticity or modulus of rigidity.*

1.5 Volumetric strain (ε_V)

The **volumetric strain** is the unit change in volume, i.e. the change in volume divided by the original volume.

1.6 Volumetric strain (ε_V) = Change in Volume/ Original Volume

A relationship similar to that for length changes holds for three-dimensional (volume) change.

- Where V is the final volume, V_0 is the original volume, and ΔV is the volume change.
- Volumetric strain is a ratio of values with the same units, so it also is a dimensionless quantity.
- $\Delta V/V$ = *volumetric strain* = $\varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$
- **Dilation:** The hydrostatic component of the total stress contributes to deformation by changing the area (or volume, in three dimensions) of an object. Area or volume change is called **dilation** and is positive or negative, as the volume increases or decreases, respectively. $e = \frac{p}{K}$ Where p is pressure.

1.7 Young's modulus or Modulus of elasticity (E) = Stress/ Strain

1.8 Modulus of rigidity or Shear modulus of elasticity (G) =

Shear Stress / Shear Strain

1.9 Bulk Modulus or Volume modulus of elasticity (K) = *Normal Stress/Volumetric stain*

1.10 Relationship between the elastic constants E, G, K, μ

$$E = 2G(1+\mu) = 3K(1-2\mu) = \frac{9KG}{3K+G}$$

[VIMP]

Where K = Bulk Modulus, μ = Poisson's Ratio, E= Young's modulus, G= Modulus of rigidity

- For a linearly elastic, isotropic and homogeneous material, the number of elastic constants required to relate stress and strain is two. i.e. any two of the four must be known.
- If the material is non-isotropic (i.e. anisotropic), then the elastic modulii will vary with additional stresses appearing since there is a coupling between shear stresses and normal stresses for an anisotropic material.

Let us take an example: The modulus of elasticity and rigidity of a material are 200 GPa and 80 GPa, respectively. Find all other elastic modulus.

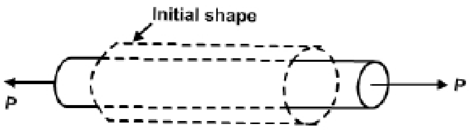
Answer: Using the relation $E = 2G(1+\mu) = 3K(1-2\mu) = \frac{9KG}{3K+G}$ we may find all other elastic modulus easily

Poisson's Ratio (μ) : $1+\mu = \frac{E}{2G}$ $\Rightarrow \mu = \frac{E}{2G} - 1 = \frac{200}{2\times 80} - 1 = 0.25$

Bulk Modulus (K) : $3K = \frac{E}{1-2\mu}$ $\Rightarrow K = \frac{E}{3(1-2\mu)} = \frac{200}{3(1-2\times 0.25)} = 133.33\text{GPa}$

1.11 Poisson's Ratio (μ)

$$= \frac{\text{Transverse strain or lateral strain}}{\text{Longitudinal strain}} = \frac{-\epsilon_y}{\epsilon_x}$$



(Under unidirectional stress in x-direction)

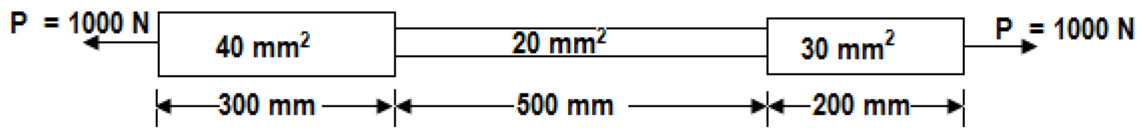
- The theory of isotropic elasticity allows Poisson's ratios in the range from -1 to 1/2.
- Poisson's ratio in various materials

Material	Poisson's ratio	Material	Poisson's ratio
Steel	0.25 – 0.33	Rubber	0.48 – 0.5
C.I	0.23 – 0.27	Cork	Nearly zero
Concrete	0.2	Novel foam	negative

- We use cork in a bottle as the cork easily inserted and removed, yet it also withstand the pressure from within the bottle. Cork with a Poisson's ratio of nearly zero, is ideal in this application.

Let us take an example: A composite rod is 1000 mm long, its two ends are 40 mm² and 30 mm² in area and length are 300 mm and 200 mm respectively. The middle portion of the rod is 20 mm² in area and 500 mm long. If the rod is subjected to an axial tensile load of 1000 N, find its total elongation. (E = 200 GPa).

Answer: Consider the following figure



Given, Load (P) =1000 N

Area; $(A_1) = 40\text{ mm}^2$, $A_2 = 20\text{ mm}^2$, $A_3 = 30\text{ mm}^2$

Length; $(l_1) = 300\text{ mm}$, $l_2 = 500\text{ mm}$, $l_3 = 200\text{ mm}$

$E = 200\text{ GPa} = 200\times 10^9\text{ N/m}^2 = 200\times 10^3\text{ N/mm}^2$

Therefore Total extension of the rod

$$\delta = \frac{P}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3}\right)$$

$$= \frac{1000N}{200\times 10^3\ N/mm^2} \times \left(\frac{300mm}{40mm^2} + \frac{500mm}{20mm^2} + \frac{200mm}{30mm^2}\right)$$

$$= 0.196\text{mm}$$

- ***Elongation of a tapered body***

Elongation of a tapering rod of length 'L' due to load 'P' at the end

- ***Elongation of a body due to its self weight***

(i) Elongation of a uniform rod of length 'L' due to its own weight 'W'

$$\delta = \frac{WL}{2AE}$$

The deformation of a bar under its own weight as compared to that when subjected to a direct axial load equal to its own weight ***will be half***.

(ii) Total extension produced in rod of length 'L' due to its own weight 'ω' per with length.

$$\delta = \frac{\omega L^2}{2EA}$$

(iii) Elongation of a conical bar due to its self weight

$$\delta = \frac{\rho g L^2}{6E} = \frac{WL}{2A_{max}E}$$

1.12 Structural members or machines must be designed such that the working stresses are less than the ultimate strength of the material.

1.13 **Factor of Safety:** (n) = $\frac{\text{Ultimate}}{\text{working}}$

1.14 Thermal or Temperature stress and strain

- When a material undergoes a change in temperature, it either elongates or contracts depending upon whether temperature is increased or decreased of the material.
- If the elongation or contraction is *not restricted*, i. e. *free* then the material does not experience *any stress despite the fact that it undergoes a strain.*
- The strain due to temperature change is called *thermal strain* and is expressed as,

$$\Delta T$$

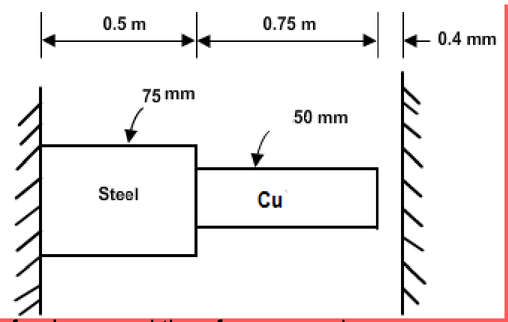
- Where α is co-efficient of thermal expansion, a material property, and ΔT is the change in temperature.
- The free expansion or contraction of materials, when restrained induces stress in the material and it is referred to as *thermal stress*.

$$\sigma_t = \alpha E(\otimes T)$$

Where, E = Modulus of elasticity

- Thermal stress produces the same effect in the material similar to that of mechanical stress. A compressive stress will produce in the material with increase in temperature and the stress developed is tensile stress with decrease in temperature.

Let us take an example: A rod consists of two parts that are made of steel and copper as shown in figure below. The elastic modulus and coefficient of thermal expansion for steel are 200 GPa and 11.7 x10^{-6} per ^{0}C respectively and for copper 70 GPa and 21.6 x 10^{-6} per ^{0}C respectively. If the temperature of the rod is raised by 50^{0}C, determine the forces and stresses acting on the rod.



Answer: If we allow this rod to freely expand then free expansion

$$\delta_T = \alpha(\Delta T)L$$
$$= (11.7\times10^{-6})\times 50\times 500 + (21.6\times10^{-6})\times 50\times 750$$
$$= 1.1025 \text{ mm (Compressive)}$$

But according to diagram only free expansion is 0.4 mm.
Therefore restrained deflection of rod =1.1025 mm – 0.4 mm = 0.7025 mm
Let us assume the force required to make their elongation vanish be P which is the reaction force at the ends.

$$\delta = \left(\frac{PL}{AE}\right)_{Steel} + \left(\frac{PL}{AE}\right)_{Cu}$$

$$\text{or } 0.7025 = \frac{P\times 500}{\left(\frac{\pi}{4}\times(0.075)^2\right)\times(200\times10^9)} + \frac{P\times 750}{\left(\frac{\pi}{4}\times(0.050)^2\right)\times(70\times10^9)}$$

$$\text{or } P = 116.6 \text{ kN}$$

Therefore, compressive stress on steel rod

$$\sigma_{Steel} = \frac{P}{A_{Steel}} = \frac{116.6\times10^3}{\frac{\pi}{4}\times(0.075)^2} \text{N/m}^2 = 26.39 \text{ MPa}$$

And compressive stress on copper rod

$$\sigma_{Cu} = \frac{P}{A_{Cu}} = \frac{116.6\times10^3}{\frac{\pi}{4}\times(0.050)^2} \text{N/m}^2 = 59.38 \text{ MPa}$$

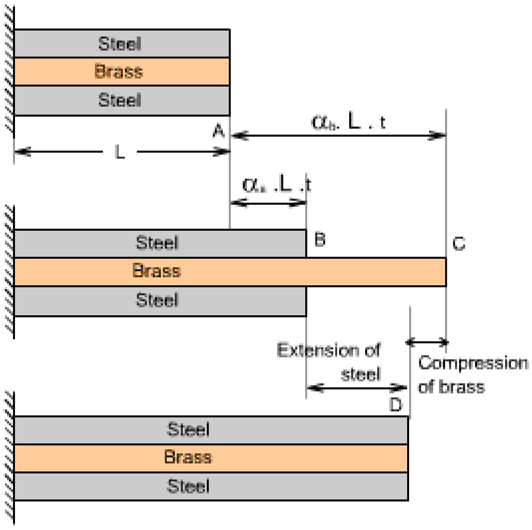
1.15 Thermal stress on Mild steel coated with Brass

A brass rod placed within a steel tube of exactly same length. The assembly is making in such a way that elongation of the combination will be same. To calculate the stress induced in the brass rod, steel tube when the combination is raised by t°C then the following analogy have to do.

(a) Original bar before heating.

(b) Expanded position if the members are allowed to expand freely and independently after heating.

(c) Expanded position of the compound bar i.e. final position after heating.



- Compatibility Equation:

$$\delta = \delta_{st} + \delta_{sf} = \delta_{Bt} - \delta_{Bf}$$

- Equilibrium Equation:

$$\sigma_s A_s = \sigma_B A_B$$

Assumption:

1. $L = L_s = L_B$
2. $\alpha_b > \alpha_s$
3. Steel → Tension

   Brass → Compression

Where, δ = Expansion of the compound bar = AD in the above figure.

δ_{st} = Free expansion of the steel tube due to temperature rise t°C = $\alpha_s L t$

= AB in the above figure.

δ_{sf} = Expansion of the steel tube due to internal force developed by the unequal expansion.

= BD in the above figure.

δ_{Bt} = Free expansion of the brass rod due to temperature rise t°C = $\alpha_b L t$

= AC in the above figure.

δ_{Bf} = Compression of the brass rod due to internal force developed by the unequal expansion.

= BD in the above figure.

And in the equilibrium equation

Tensile force in the steel tube = Compressive force in the brass rod Where,

σ_s = Tensile stress developed in the steel tube.

σ_B = Compressive stress developed in the brass rod.

A_s = Cross section area of the steel tube.

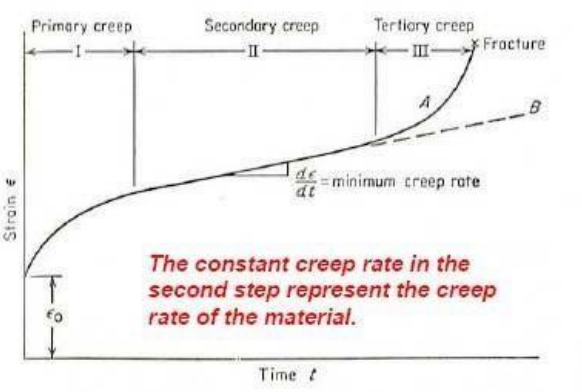
A_B = Cross section area of the brass rod.

1.18 Creep

When a member is subjected to a constant load over a long period of time it undergoes a slow permanent deformation and this is termed as "creep". This is dependent on temperature. Usually at elevated temperatures creep is high.

- The materials have its own different melting point; each will creep when the homologous temperature > 0.5. Homologous temp = $\frac{\text{Testing temperature}}{\text{Melting temperature}} > 0.5$

A typical creep curve shows three distinct stages with different creep rates. After an initial rapid elongation ε_o, the creep rate decrease with time until reaching the steady state.



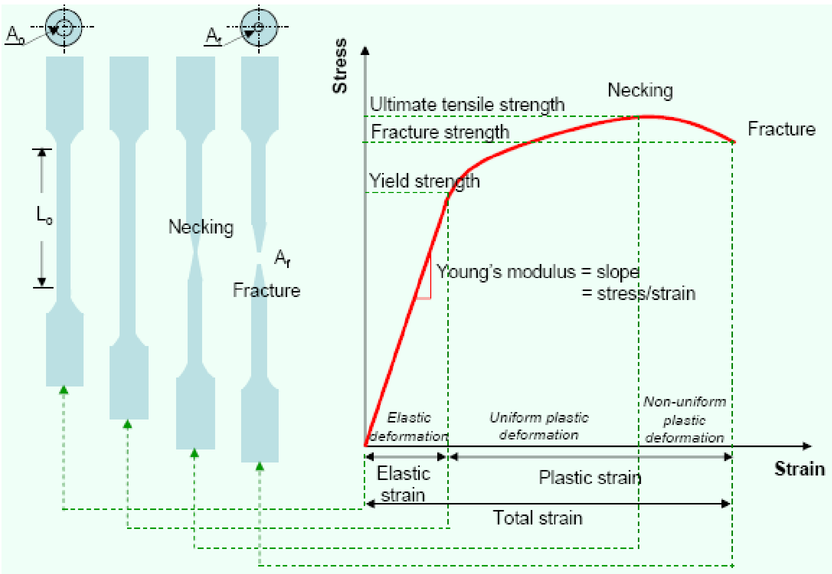
1) ***Primary creep*** is a period of transient creep. The creep resistance of the material increases due to material deformation.

2) ***Secondary creep*** provides a nearly constant creep rate. The average value of the creep rate during this period is called the minimum creep rate. A stage of balance between competing ***strain hardening*** and ***recovery*** (softening) of the material.

3) ***Tertiary creep*** shows a rapid increase in the creep rate due to effectively reduced cross-sectional area of the specimen leading to *creep rupture* or failure. In this stage *intergranular* cracking and/or formation of voids and cavities occur.

$$\frac{PL}{A_x E_x \square A_y}$$
$$E_y$$

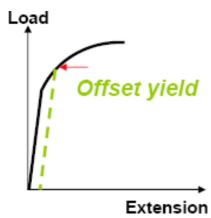
1.19 Tension Test



i) True elastic limit: based on micro-strain measurement at strains on order of 2 x 10^{-6}. Very low value and is related to the motion of a few hundred dislocations.

ii) Proportional limit: the highest stress at which stress is directly proportional to strain.

iii) Elastic limit: is the greatest stress the material can withstand without any measurable permanent strain after unloading. Elastic limit > proportional limit.



iv) Yield strength is the stress required to produce a small specific amount of deformation.

The offset yield strength can be determined by the stress corresponding to the intersection of the stress-strain curve and a line parallel to the elastic line offset by a strain of 0.2 or 0.1%.

(□0.002 or 0.001)

- The offset yield stress is referred to proof stress either at 0.1 or 0.5% strain used for design and specification purposes to avoid the practical difficulties of measuring the elastic limit or proportional limit.

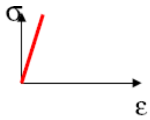
v) Tensile strength or ultimate tensile strength (UTS) $\square_u$ is the maximum load P_{max} divided by the original cross-sectional area A_o of the specimen.

vi) % Elongation, $\square \frac{L_f \square L_o}{L_o}$, is chiefly influenced by uniform elongation, which is dependent on the strain-hardening capacity of the material.

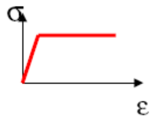
vii) **Reduction of Area:** $q = \frac{A_o - A_f}{A_o}$

- Reduction of area is more a measure of the deformation required to produce failure and its chief contribution results from the necking process.
- Because of the complicated state of stress state in the neck, values of reduction of area are dependent on specimen geometry, and deformation behaviour, and they should not be taken as true material properties.
- RA is the most structure-sensitive ductility parameter and is useful in detecting quality changes in the materials.

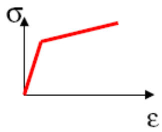
viii) Stress-strain response



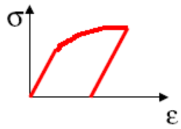
Linear elastic



Linear elastic-perfectly plastic



Linear elastic-hardening plastic



Linear elastic-hardening plasticity with unloading

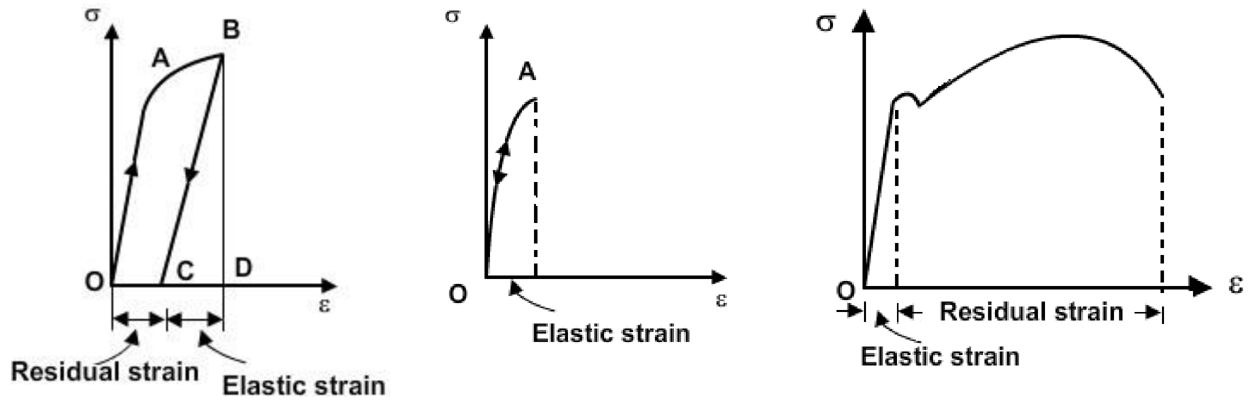
1.20 Elastic strain and Plastic strain

The strain present in the material after unloading is called the **residual strain or plastic strain** and the strain disappears during unloading is termed as **recoverable or elastic strain.**

Equation of the straight line CB is given by

$\sigma = \varepsilon_{total} E - \varepsilon_{Plastic} E = \varepsilon_{Elastic} E$

Carefully observe the following figures and understand which one is Elastic strain and which one is Plastic strain

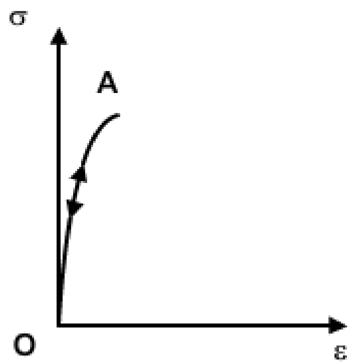


Let us take an example: A 10 mm diameter tensile specimen has a 50 mm gauge length. The load corresponding to the 0.2% offset is 55 kN and the maximum load is 70 kN. Fracture occurs at 60 kN. The diameter after fracture is 8 mm and the gauge length at fracture is 65 mm. Calculate the following properties of the material from the tension test.

(i) % Elongation
(ii) Reduction of Area (RA) %
(iii) Tensile strength or ultimate tensile strength (UTS)
(iv) Yield strength
(v) Fracture strength
(vi) If E = 200 GPa, the elastic recoverable strain at maximum load
(vii) If the elongation at maximum load (the uniform elongation) is 20%, what is the plastic strain at maximum load?

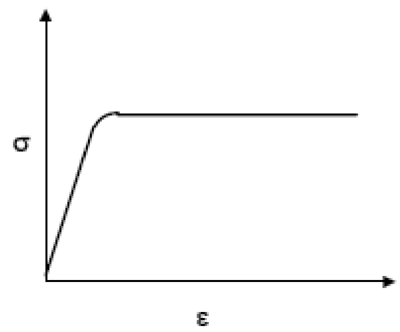
1.21 Elasticity

This is the property of a material to regain its original shape after deformation when the external forces are removed. When the material is in elastic region the strain disappears completely after removal of the load, The stress-strain relationship in elastic region need not be linear and can be non-linear (example rubber). The maximum stress value below which the strain is fully recoverable is called the *elastic limit*. It is represented by point A in figure. All materials are elastic to some extent but the degree varies, for example, both mild steel and rubber are elastic materials but steel is more elastic than rubber.



1.22 Plasticity

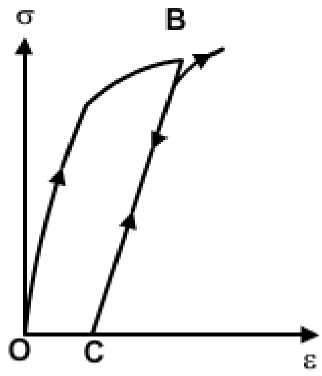
When the stress in the material exceeds the elastic limit, the material enters into plastic phase where the strain can no longer be completely removed. Under plastic conditions materials ideally deform without any increase in stress. A typical stress strain diagram for an elastic-perfectly plastic



material is shown in the figure. Mises-Henky criterion gives a good starting point for plasticity analysis.

1.23 Strain hardening

If the material is reloaded from point C, it will follow the previous unloading path and line CB becomes its new elastic region with elastic limit defined by point B. Though the new elastic region CB resembles that of the initial elastic region OA, the internal structure of the material in the new state has changed. The change in the microstructure of the material is clear from the fact that the ductility of the material has come down due to strain hardening. When the material is reloaded, it follows the same path as that of a virgin material and fails on reaching the ultimate strength which remains unaltered due to the intermediate loading and unloading process.



Question:

1. Define Hook's law.
2. Explain the stress and strain curve for the ductile material.
3. Define the following terms:
   (a) Modulus of elasticity (b) principal stresses (c) proof stress
4. A 10 mm diameter tensile specimen has a 50 mm gauge length. The load corresponding to the 0.2% offset is 55 kN and the maximum load is 70 kN. Fracture occurs at 60 kN. The diameter after fracture is 8 mm and the gauge length at fracture is 65 mm. Calculate the following properties of the material from the tension test.
   % Elongation
   Reduction of Area (RA) %
   Tensile strength or ultimate tensile strength (UTS)
   Yield strength
   Fracture strength
   If E = 200 GPa, the elastic recoverable strain at maximum load
   If the elongation at maximum load (the uniform elongation) is 20%, what is the plastic strain at maximum load?
5. What is FOS?
6. Write the properties of material.

Chapter No. 2

RESILIENCE (STRAIN ENERGY)

2.1 Introduction: -

Strain energy is as the energy which is stored within a material when work has been done on the material. Here it is assumed that the material remains elastic whilst work is done on it so that all the energy is recoverable and no permanent deformation occurs due to yielding of the material,

Strain energy U = work done

Thus for a gradually applied load the work done in straining the material will be given by the shaded area under the load-extension graph of Fig.

$$U = \frac{1}{2} P \delta$$

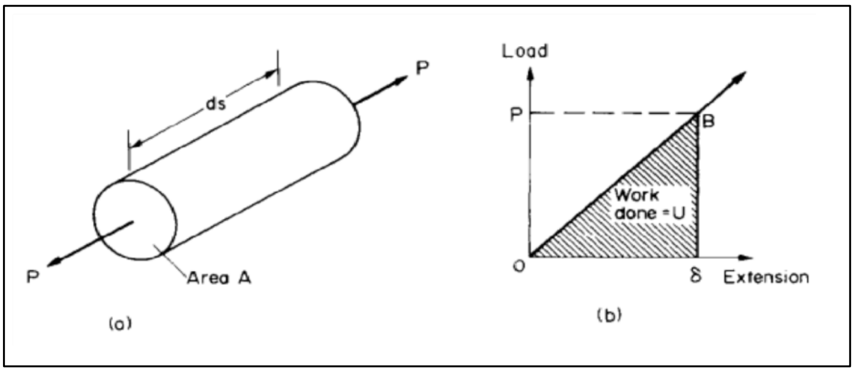


Figure 2.1: - Work done by a gradually applied load.

The unshaded area above the line OB of Fig. 2.1 is called the complementary energy, a quantity which is utilized in some advanced energy methods of solution and is not considered within the terms of reference of this text.

Resilience

The capability of a strained body to recover its size and shape after deformation caused especially by compressive stress or external load

Proof Resilience is defined as the maximum energy that can be absorbed up to the elastic limit, without creating a permanent distortion.

Modulus of resilience is the maximum amount of energy per volume that a material can absorb and still recover.

2.2 Strain energy –Due to Gradually Applied Load Neglecting the weight of the bar: -

Consider a small element of a bar, length ds, shown in Fig. 7.1. If a graph is drawn of load against elastic extension the shaded area under the graph gives the work done and hence the strain energy,

$U = \frac{1}{2} P \delta$ ….(1)

But young modulus $E = \frac{Pds}{A\delta}$ $\therefore \ \delta = \frac{Pds}{AE}$ …. (2)

Now, substituting eqn. (2) in (1) For

bar element, $U = \frac{P^2 ds}{2AE}$

$\therefore$ Total strain energy for a bar of length L, $U = \int_o^L (\frac{P^2 ds}{2AE})$

b- Including the weight of the bar: -

Consider now a bar of length L mounted vertically, as shown in Fig. 7.2. At any section A B the total load on the section will be the external load P together with the weight of the bar material below AB.

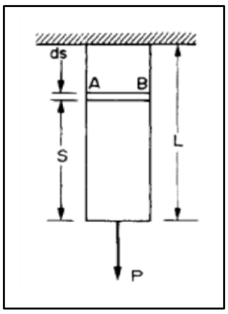


Figure 2.2: - Direct load - tension or compression.

Load on section *A B* = $P \pm \rho g A s$

The positive sign being used when *P* is tensile and the negative sign when *P* is compressive. Thus, for a tensile force *P* the extension of the element *ds* is given by the definition of Young's modulus *E* to be

$$\delta = \frac{\sigma ds}{E}$$

$$\delta = \frac{(P \pm \rho g A s) ds}{AE}$$

But work done = $\frac{1}{2}$ load x extension

$$= \frac{1}{2}(P \pm \rho g A s)\frac{(P \pm \rho g A s) ds}{AE}$$

$$= \frac{P^2}{2AE}ds + \frac{P\rho g}{E}sds + \frac{(\rho g)^2 A}{2E}s^2 ds$$

So, the total strain energy or work done is,

$$U = \int_0^L \frac{P^2}{2AE}ds + \int_0^L \frac{P\rho g}{E}sds + \int_0^L \frac{(\rho g)^2 A}{2E}s^2 ds$$

….1.2

1-2 Strain energy-shear: -

Consider the elemental bar now subjected to a shear load Q at one end causing deformation through the angle y (the shear strain) and a shear deflection 6, as shown in Fig. 1.3.

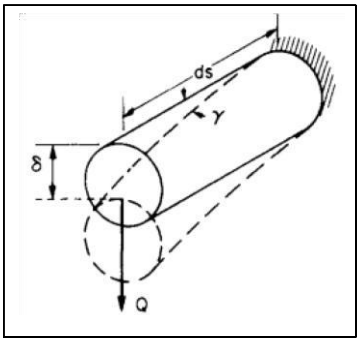


Figure 1.3: - Shear.

Strain energy U = work done = $\frac{1}{2}Q\delta = \frac{1}{2}Q\gamma ds$ …. (1) But

modulus of rigidity $G = \frac{\tau}{\gamma} = \frac{Q}{\gamma A}$

$$\gamma = \frac{Q}{AG}$$

Substitute eqn. (2) in (1),

$$U = \frac{1}{2}Q\frac{Q}{AG}ds$$

Shear strain energy $= \frac{Q^2}{2AG}ds$

:. Total strain energy resulting from shear $= \int_0^L \frac{Q^2}{2AG}ds$

QUESTIONS:

1. What is the strain energy caused due to self weight in a cylindrical bar?
2. What is the maximum stress induced in a bar 2500 mm^2, when a load of 2000 kN is applied suddenly?
3. Load is gradually applied on a steel plate shown below, what is the modulus of resilience, if 130 N/mm^2 is the maximum stress applied? (Assume E = 100 x 10^3 Mpa)

4. What is the proof resilience of a square bar of 2500 mm^2 and 200 mm long, when a load of 150 kN is induced gradually? (Take E = 150 x 10^3 Mpa)

Energy stored

Chapter 3
Moment of Inertia and Centroid

1.10 Centre of gravity: The centre of gravity of a body defined as the point through which the whole weight of a body may be assumed to act.

1.11 Centroid or Centre of area: The centroid or centre of area is defined as the point where the whole area of the figure is assumed to be concentrated.

1.12 Moment of Inertia (MOI)

- About any point the product of the force and the perpendicular distance between them is known as moment of a force or first moment of force.
- This first moment is again multiplied by the perpendicular distance between them to obtain second moment of force.
- In the same way if we consider the area of the figure it is called second moment of area or area moment of inertia and if we consider the mass of a body it is called second moment of mass or mass moment of Inertia.
- Mass moment of inertia is the measure of resistance of the body to rotation and forms the basis of dynamics of rigid bodies.
- Area moment of Inertia is the measure of resistance to bending and forms the basis of strength of materials.

1.13 Mass moment of Inertia (MOI)

$$I = \sum_i m_i r_i^2$$

3 Notice that the moment of inertia 'I' depends on the distribution of mass in the system.

4 The further the mass is from the rotation axis, the bigger the moment of inertia.

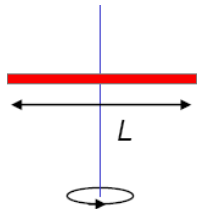
5 For a given object, the moment of inertia depends on where we choose the rotation axis.

6 In rotational dynamics, the moment of inertia 'I' appears in the same way that mass *m* does in linear dynamics.

7 *Solid disk or cylinder of mass M and radius R,* about perpendicular axis through its centre, $I = \frac{1}{2} MR^2$

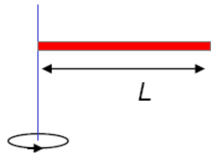
8 Solid sphere of mass M and radius R, about an axis through its centre, $I = 2/5\, MR^2$

9 Thin *rod of mass M and length L*, about a perpendicular axis through its centre.



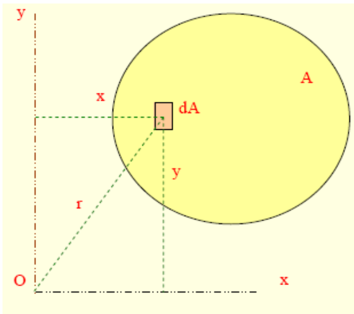
$$I = \frac{1}{12} ML^2$$

10 Thin rod of mass M and length L, about a perpendicular axis through its end.



$$I = \frac{1}{3} ML^2$$

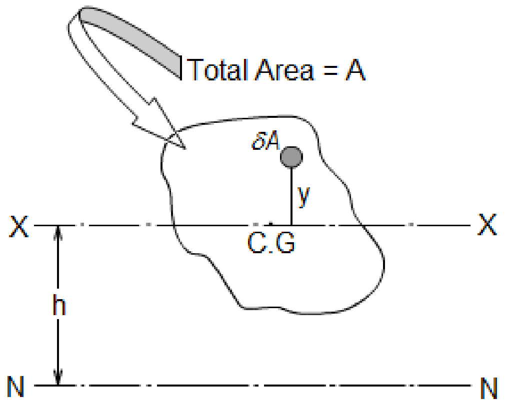
1.14 Area Moment of Inertia (MOI) or Second moment of area



- To find the centroid of an area by the first moment of the area about an axis was determined ($\int x\, dA$)
- Integral of the ***second moment of area*** is called moment of inertia ($\int x^2 dA$)
- Consider the area (A)
- By definition, the moment of inertia of the differential area about the x and y axes are dI_{xx} and dI_{yy}
- $dI_{xx} = y^2 dA \Rightarrow$ **$I_{xx} = \int y^2 dA$**
- $dI_{yy} = x^2 dA \Rightarrow$ **$I_{yy} = \int x^2\, dA$**

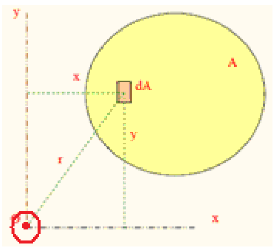
1.15 Parallel axis theorem for an area: The rotational inertia about any axis is the sum of second moment of inertia about a parallel axis through the C.G and total area of the body times square of the distance between the axes.

$I_{NN} = I_{CG} + Ah^2$



1.16 **Perpendicular axis theorem for an area:** If x, y & z are mutually perpendicular axes as shown, then $I_{zz}(J) = I_{xx} + I_{yy}$

Z-axis is perpendicular to the plane of x-y and vertical to this page as shown in figure.



- To find the moment of inertia of the differential area about the pole (point of origin) or z-axis, **(r)** is used. (r) is the perpendicular distance from the pole to dA for the entire area

$\mathbf{J = \int r^2\, dA = \int (x^2 + y^2)\,dA = I_{xx} + I_{yy}}$ (since $r^2 = x^2 + y^2$) Where, J = polar moment of inertia

1.17 Moments of Inertia (area) of some common area:

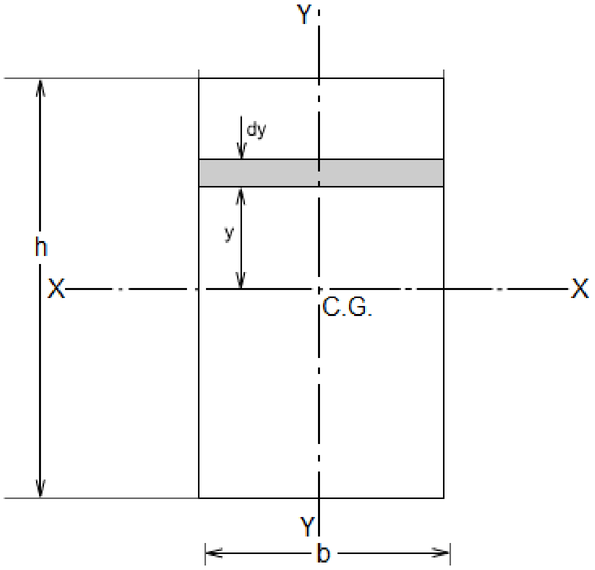
2 MOI of Rectangular area

Moment of inertia about axis XX which passes through centroid.

Take an element of width 'dy' at a distance y from XX axis.

∴ Area of the element (dA) = b× dy.

and Moment of Inertia of the element about XX axis $= dA \times y^2 = b.y^2.dy$



∴Total MOI about XX axis (Note it is area moment of Inertia)

$$I_{xx} = \int_{-h/2}^{+h/2} by^2dy = 2b\int_0^{h/2} y^2dy = \frac{bh^3}{12}$$

$$I_{xx} = \frac{bh^3}{12}$$

Similarly, we may find, $I_{yy} = \frac{hb^3}{12}$

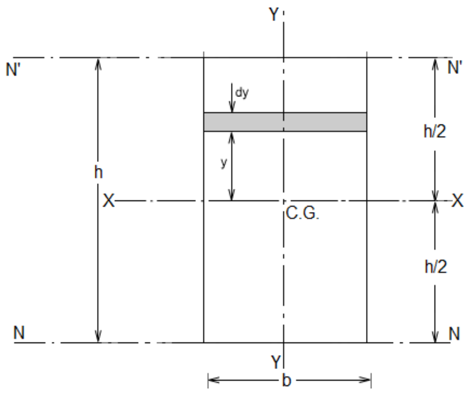
$\therefore$ Polar moment of inertia (J) = $I_{xx} + I_{yy} = \frac{bh^3}{12} + \frac{hb^3}{12}$

If we want to know the MOI about an axis NN passing through the bottom edge or top edge.

Axis XX and NN are parallel and at a distance h/2.

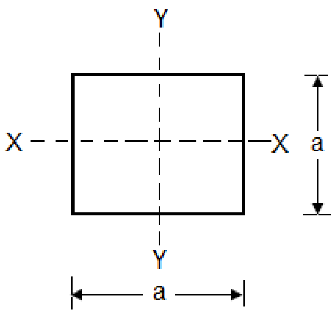
Therefore $I_{NN} = I_{xx} +$ Area $\times$ (distance) ^{2}

$$= \frac{bh^3}{12} + b \times h \times \left(\frac{h}{2}\right)^2 = \frac{bh^3}{3}$$



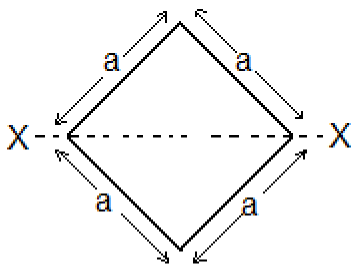
Case-I: Square area

a^4 $I_{xx} =$

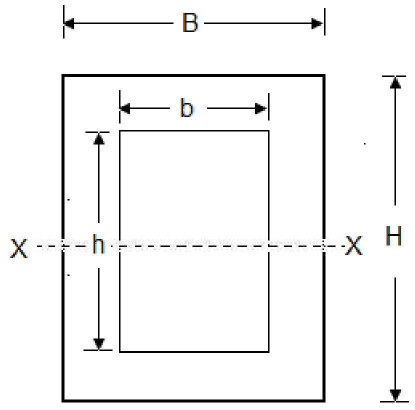


Case-II: Square area with diagonal as axis

a^4 $I_{xx} =$



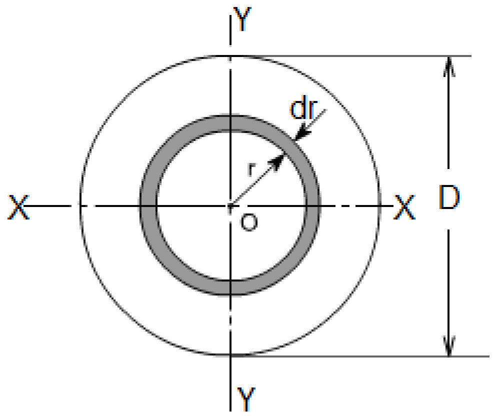
Case-III: **Rectangular area with a centrally rectangular hole**



Moment of inertia of the area = moment of inertia of BIG rectangle – moment of inertia of SMALL rectangle

$$I_{xx} = \frac{BH^3}{12} - \frac{bh^3}{12}$$

3 MOI of a Circular area



The moment of inertia about axis XX this passes through the centroid. It is very easy to find polar moment of inertia about point 'O'. Take an element of width 'dr' at a distance 'r' from centre. Therefore, the moment of inertia of this element about polar axis

$d(J) = d(I_{xx} + I_{yy}) = \text{area of ring} \times (\text{radius})^2$

or $d(J) = 2\pi r dr \times r^2$

Integrating both side we get

$$J = \int_0^R 2\pi r^3 dr = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}$$

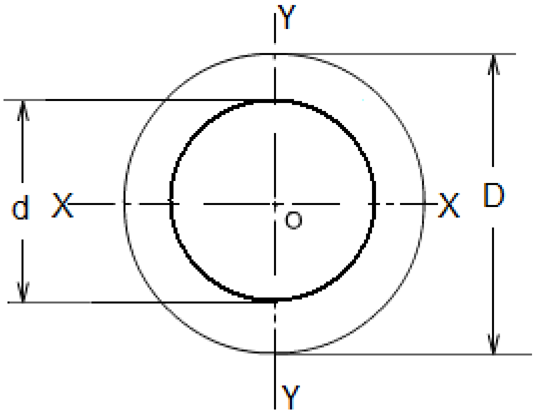
Due to summetry $I_{xx} = I_{yy}$

Therefore, $I_{xx} = I_{yy} = \frac{J}{2} = \frac{\pi D^4}{64}$

$$I_{xx} = I_{yy} = \frac{\pi D^4}{32}$$

Case–I: Moment of inertia of a circular area with a concentric hole.

Moment of inertia of the area = moment of inertia of BIG circle – moment of inertia of SMALL circle.

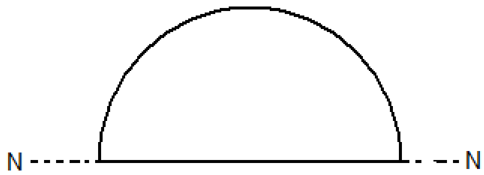


$$I_{xx} = I_{yy} = \frac{\pi D^4}{64} - \frac{\pi d^4}{64}$$

$$= \frac{\pi}{64}(D^4 - d^4)$$

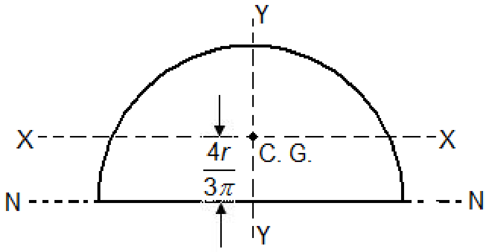
and $J = \frac{\pi}{32}(D^4 - d^4)$

Case–II: Moment of inertia of a semi-circular area.



$I_{NN} = \frac{1}{2}$ of the momemt of total circular lamina

$$= \frac{1}{2} \times \left(\frac{\pi D^4}{64}\right) = \frac{\pi D^4}{128}$$



We know that distance of CG from base is

$$\frac{4r}{3\pi} = \frac{2D}{3\pi} = h(\text{say})$$

i.e. distance of parallel axis XX and NN is (h)

∴ According to parallel axis theory

$$I_{NN} = I_G + \text{Area} \times (\text{distance})^2$$

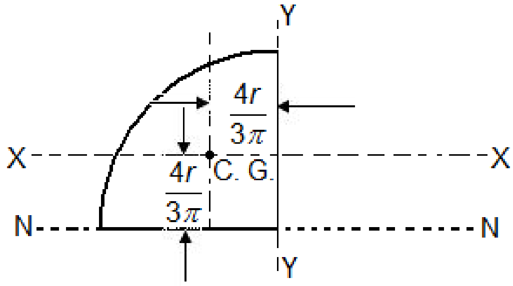
or $\frac{\pi D^4}{128} = I_{xx} + \frac{1}{2}\left(\frac{\pi D^2}{4}\right) \times (h)^2$

or $\frac{\pi D^4}{128} = I_{xx} + \frac{1}{2} \times \left(\frac{\pi D^2}{4}\right) \times \left(\frac{2D}{3\pi}\right)^2$ (i)

or $I_{xx} = 0.11R^4$

Case – III: **Quarter circle area**

I_{XX} = one half of the moment of Inertia of the Semi-circular area about XX.



$$I_{xx} = \frac{1}{2} \times \left(0.11R^4\right) = 0.055\ R^4$$

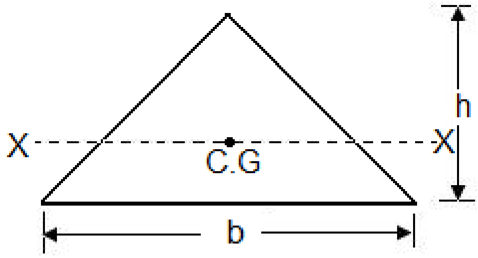
$$I_{XX} = 0.055R^4$$

I_{NN} = one half of the moment of Inertia of the Semi-circular area about NN.

$$\therefore I_{NN} = \frac{1}{2} \times \frac{\pi D^4}{64} = \frac{\pi D^4}{128}$$

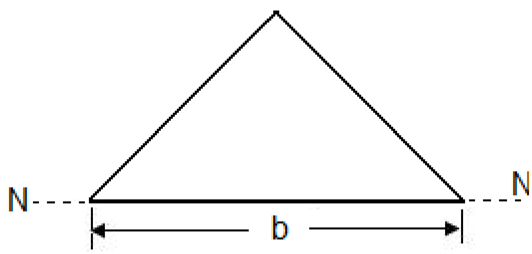
4 Moment of Inertia of a Triangular area

- **Moment of Inertia of a Triangular area of a axis XX parallel to base and passes through C.G.**



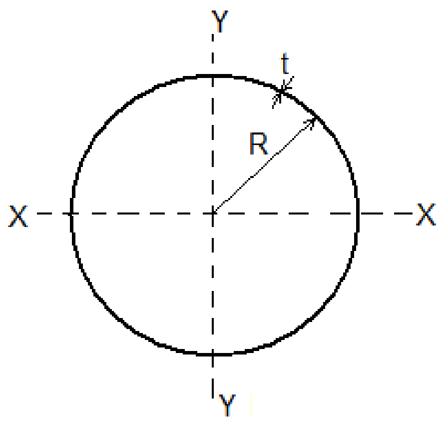
$$I_{XX} = \frac{bh^3}{36}$$

- **Moment of inertia of a triangle about an axis passes through base**



$$I_{NN} = \frac{bh^3}{12}$$

5 Moment of inertia of a thin circular ring



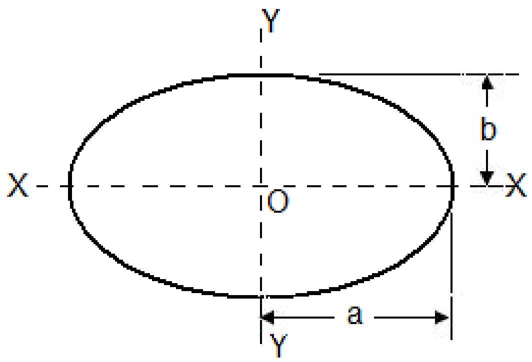
Polar moment of Inertia

$(J) = R^2 \times$ area of whole ring

$= R^2 \times 2\pi Rt = 2\pi R^3 t$

$$I_{XX} = I_{YY} = \frac{J}{2} = \pi R^3 t$$

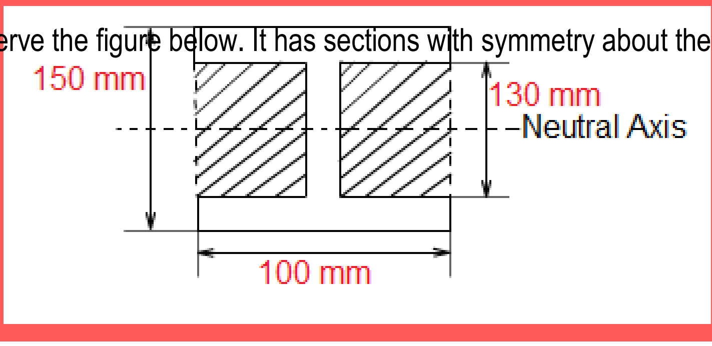
6 Moment of inertia of a elliptical area



$$I_{XX} = \frac{\pi a b^3}{4}$$

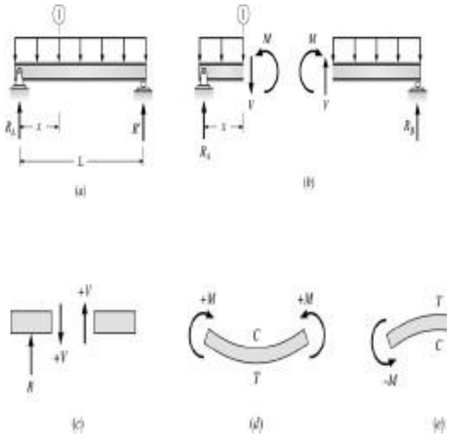
Let us take an example: An I-section beam of 100 mm wide, 150 mm depth flange and web of thickness 20 mm is used in a structure of length 5 m. Determine the Moment of Inertia (of area) of cross-section of the beam.

Answer: Carefully observe the figure below. It has sections with symmetry about the neutral axis.



Chapter - 4 Bending Moment and Shear Force Diagram

• Bending Moment and Shear Force Diagram



Shear Force (V) ≡ equal in magnitude but opposite in direction to the algebraic sum (resultant) of the components in the direction perpendicular to the axis of the beam of all external loads and support reactions acting on either side of the section being considered.

Bending Moment (M) ≡ equal in magnitude but opposite in direction to the algebraic sum of the moments about (the centroid of the cross section of the beam) the section of all external loads and support reactions acting on either side of the section being considered.

What are the benefits of drawing shear force and bending moment diagram?

The benefits of drawing a variation of shear force and bending moment in a beam as a function of 'x' measured from one end of the beam is that it becomes easier to determine the maximum absolute value of shear force and bending moment. The shear force and bending moment diagram gives a clear picture in our mind about the variation of SF and BM throughout the entire section of the beam.

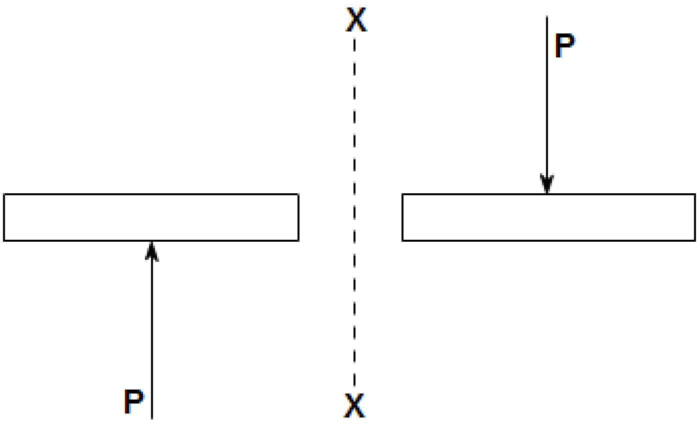
Further, the determination of value of bending moment as a function of 'x' becomes very important so as to determine $\frac{d^2}{dx^2}$ x

• Notation and sign convention

• Shear force (V)

Positive Shear Force

A shearing force having a downward direction to the right hand side of a section or upwards to the left hand of the section will be taken as 'positive'. It is the usual sign conventions to be followed for the shear force. In some book followed totally opposite sign convention.

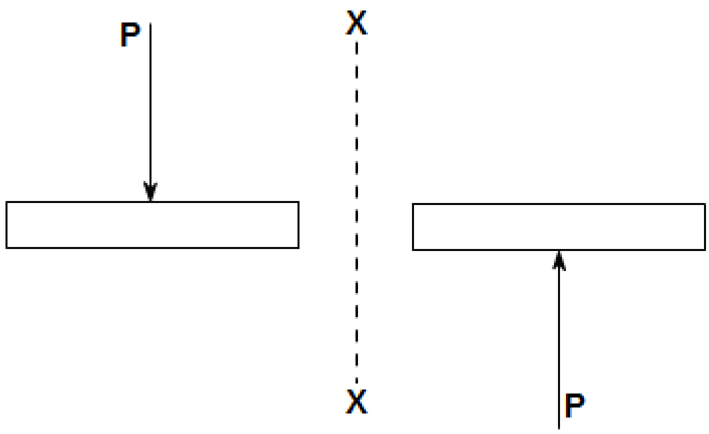


The upward direction shearing force which is on the left hand of the section XX is positive shear force.

The downward direction shearing force which is on the right hand of the section XX is positive shear force.

Negative Shear Force

A shearing force having an upward direction to the right hand side of a section or downwards to the left hand of the section will be taken as 'negative'.



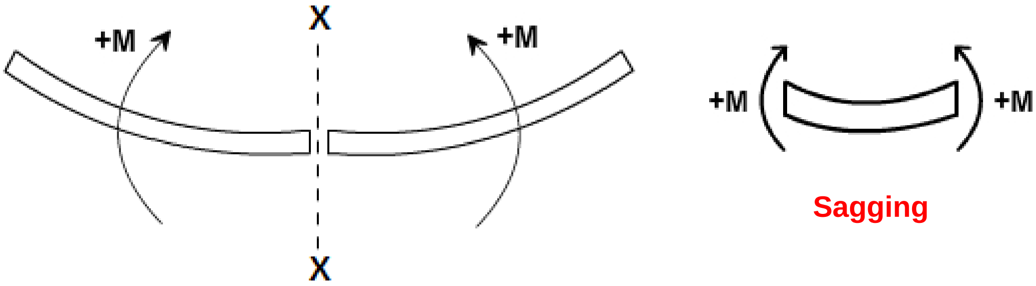
The downward direction shearing force which is on the left hand of the section XX is negative shear force.

The upward direction shearing force which is on the right hand of the section XX is negative shear force.

- **Bending Moment (M)**

Positive Bending Moment

A bending moment causing concavity upwards will be taken as 'positive' and called as sagging bending moment.

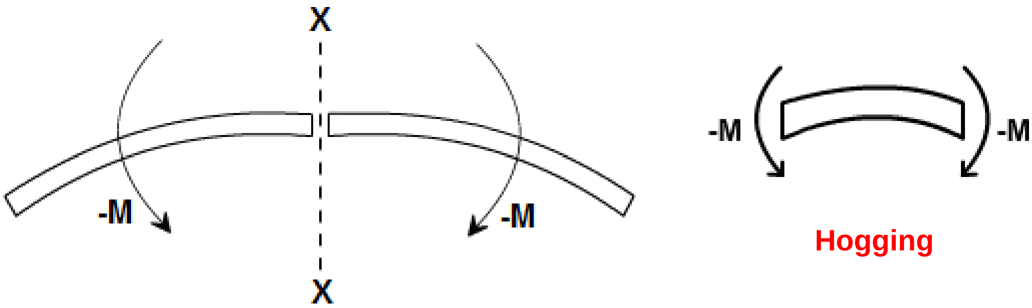


If the bending moment of the left hand of the section XX is clockwise then it is a positive bending moment.

If the bending moment of the right hand of the section XX is anti- clockwise then it is a positive bending moment.

A bending moment causing concavity upwards will be taken as 'positive' and called as sagging bending moment.

Negative Bending Moment



If the bending moment of the left hand of the section XX is anti- clockwise then it is a positive bending moment.

If the bending moment of the right hand of the section XX is clockwise then it is a positive bending moment.

A bending moment causing convexity upwards will be taken as 'negative' and called as hogging bending moment.

Way to remember sign convention

- **Remember** in the ***Cantilever beam both Shear force and BM are negative (–ive).***

- **Relation between S.F (V_x), B.M. (M_x) & Load (w)**

- $\frac{dV_x}{dx} = -w\ (load)$ The value of the distributed load at any point in the beam is

equal to the slope of the shear force curve. (Note that the sign of this rule may change depending on the sign convention used for the external distributed load).

- $\frac{dM_x}{dx} = V_x$ The value of the shear force at any point in the beam is equal to the slope of the bending moment curve.

- **Procedure for drawing shear force and bending moment diagram**

Construction of shear force diagram

- From the loading diagram of the beam constructed shear force diagram.
- First determine the reactions.
- Then the vertical components of forces and reactions are successively summed from the left end of the beam to preserve the mathematical sign conventions adopted. The shear at a section is simply equal to the sum of all the vertical forces to the left of the section.
- The shear force curve is continuous unless there is a point force on the beam. The curve then "jumps" by the magnitude of the point force (+ for upward force).
- When the successive summation process is used, the shear force diagram should end up with the previously calculated shear (reaction at right end of the beam). No shear

force acts through the beam just beyond the last vertical force or reaction. If the shear force diagram closes in this fashion, then it gives an important check on mathematical calculations. i.e. The shear force will be zero at each end of the beam unless a point force is applied at the end.

Construction of bending moment diagram

- The bending moment diagram is obtained by proceeding continuously along the length of beam from the left hand end and summing up the areas of shear force diagrams using proper sign convention.
- The process of obtaining the moment diagram from the shear force diagram by summation is exactly the same as that for drawing shear force diagram from load diagram.

- The bending moment curve is continuous unless there is a point moment on the beam. The curve then "jumps" by the magnitude of the point moment (+ for CW moment).
- We know that a constant shear force produces a uniform change in the bending moment, resulting in straight line in the moment diagram. If no shear force exists along a

certain portion of a beam, then it indicates that there is no change in moment takes place. We also know that dM/dx= V_x therefore, from the fundamental theorem of calculus the maximum or minimum moment occurs where the shear is zero.

- The bending moment will be zero at each free or pinned end of the beam. If the end is built in, the moment computed by the summation must be equal to the one calculated initially for the reaction.

• Different types of Loading and their S.F & B.M Diagram

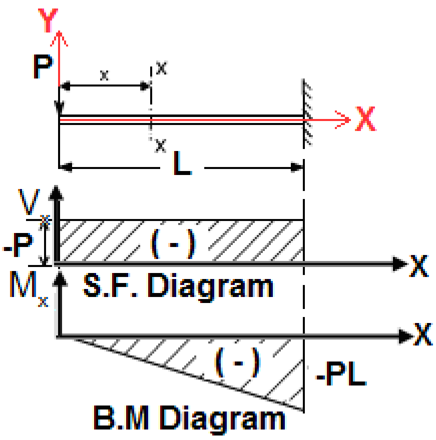
4. A Cantilever beam with a concentrated load 'P' at its free end.

Shear force:

At a section a distance x from free end consider the forces to the left, then $(V_x) = - P$ (for all values of x) negative in sign i.e. the shear force to the left of the x-section are in downward direction and therefore negative.

Bending Moment:

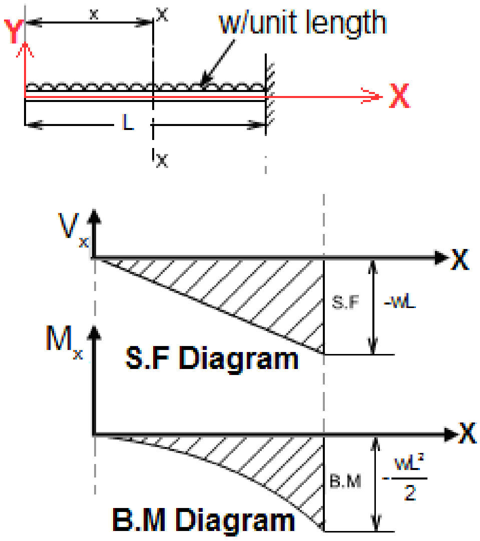
Taking moments about the section gives (obviously to the left of the section) $M_x = -P.x$ (negative sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as negative according to the sign convention) so that the maximum bending moment occurs at the fixed end i.e. $M_{max} = - PL$ (at x = L)



S.F and B.M diagram

5. A Cantilever beam with uniformly distributed load over the whole length

When a cantilever beam is subjected to a uniformly distributed load whose intensity is given w /unit length.



S.F and B.M diagram

Shear force:

Consider any cross-section XX which is at a distance of x from the free end. If we just take the resultant of all the forces on the left of the X-section, then

$V_x = -w.x$ for all values of 'x'.

At x = 0, $V_x = 0$

At x = L, $V_x = -wL$ (i.e. Maximum at fixed end)

Plotting the equation **$V_x = -w.x$**, we get a straight line because it is a equation of a straight line y (V_x) = m(- w) .x **Bending Moment:**

Bending Moment at XX is obtained by treating the load to the left of XX as a concentrated load of the same value (w.x) acting through the centre of gravity at x/2.

Therefore, the bending moment at any cross-section XX is

$$M_x = (-w.x).\frac{x}{2} = -\frac{w.x^2}{2}$$

Therefore *the variation of bending moment is according to parabolic law.*

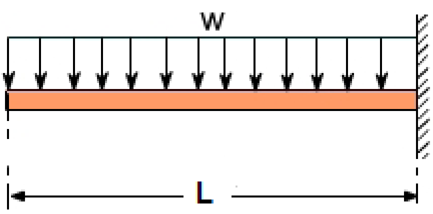
The extreme values of B.M would be at

x = 0, $M_x = 0$

and x = L, $M_x = -\frac{wL^2}{2}$

Maximum bending moment, $M_{max} = \frac{wL^2}{2}$ at fixed end

Another way to describe a cantilever beam with uniformly distributed load (UDL) over it's whole length.



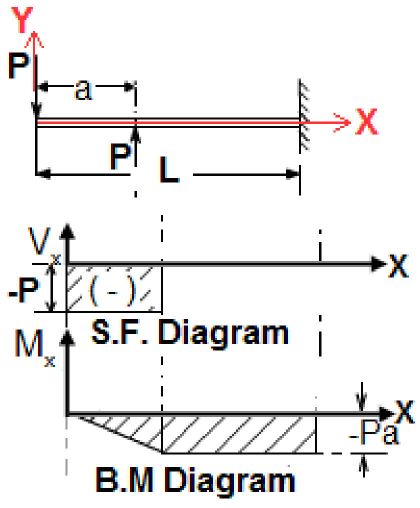
6. **A Cantilever beam loaded as shown below draw its S.F and B.M diagram**

In the region 0 < x < a

Following the same rule as followed previously, we get V_x =- P; and M_x = - P.x

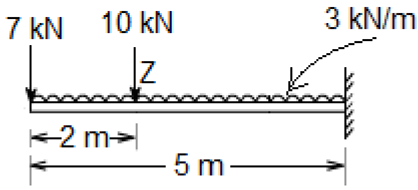
In the region a < x < L

V_x =- P+P=0; and M_x = - P.x +P$(x-a) = P.a$



S.F and B.M diagram

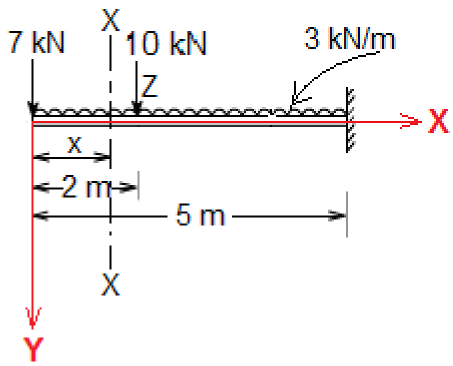
7. ***Let us take an example:*** Consider a cantilever bean of 5 m length. It carries a uniformly distributed load 3 KN/m and a concentrated load of 7 kN at the free end and 10 kN at 3 meters from the fixed end.



Draw SF and BM diagram.

Answer: In the region 0 < x < 2 m

Consider any cross section XX at a distance x from free end.



Shear force (V_x) = -7- 3x

So, the variation of shear force is linear. at x = 0, V_x = -7kN

at x = 2 m , V_x = -7 - 3 × 2 = -13 kN

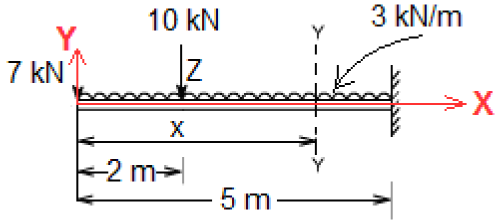
at point Z V_x = -7 -3× 2-10 = -23 kN

Bending moment (M_x) $= -7x - (3x).\frac{x}{2} = -\frac{3x^2}{2} - 7x$

So, the variation of bending force is parabolic. at x = 0, $M_x = 0$

at x = 2 m, $M_x = -7 \times 2 - (3\times 2) \times \frac{2}{2} = -20$ kNm

In the region 2 m < x < 5 m



Consider any cross section YY at a distance x from free end

Shear force (V_x) = -7 - 3x – 10 = -17- 3x So, the variation of shear force is linear. at x = 2 m, V_x = - 23 kN

at x = 5 m, V_x = - 32 kN

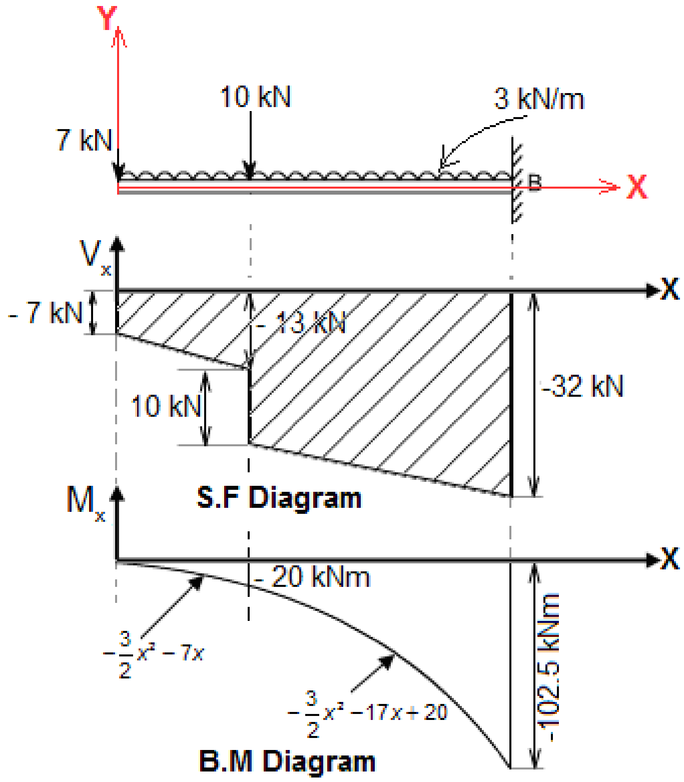
Bending moment (M_x) = $-7x - (3x) \times \left(\frac{x}{2}\right) - 10(x-2)$

$$= -\frac{3}{2}x^2 - 17x + 20$$

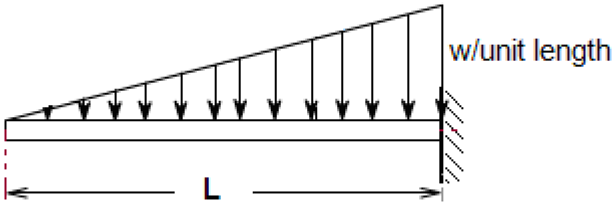
So, the variation of bending force is parabolic.

at x = 2 m, $M_x = -\frac{3}{2} \times 2^2 - 17 \times 2 + 20$ = - 20 kNm

at x = 5 m, M_x = - 102.5 kNm



8. A Cantilever beam carrying uniformly varying load from zero at free end and w/unit length at the fixed end



Consider any cross-section XX which is at a distance of x from the free end. At this point load (w_x) = $\frac{w}{L}.x$

Therefore total load (W) $= \int_0^L w_x dx = \int_0^L \frac{w}{L}.xdx = \frac{wL}{2}$

Shear force $(V_x) =$ area of ABC (load triangle)

$= -\frac{1}{2}.\left(\frac{w}{L}x\right).x = -\left(\frac{wx^2}{2L}\right)$

∴ The shear force variation is parabolic.

at x = 0, $V_x = 0$

at x = L, $V_x = -\frac{WL}{2}$ i.e. Maximum Shear force $(V_{max}) = \frac{-WL}{2}$ at fixed end

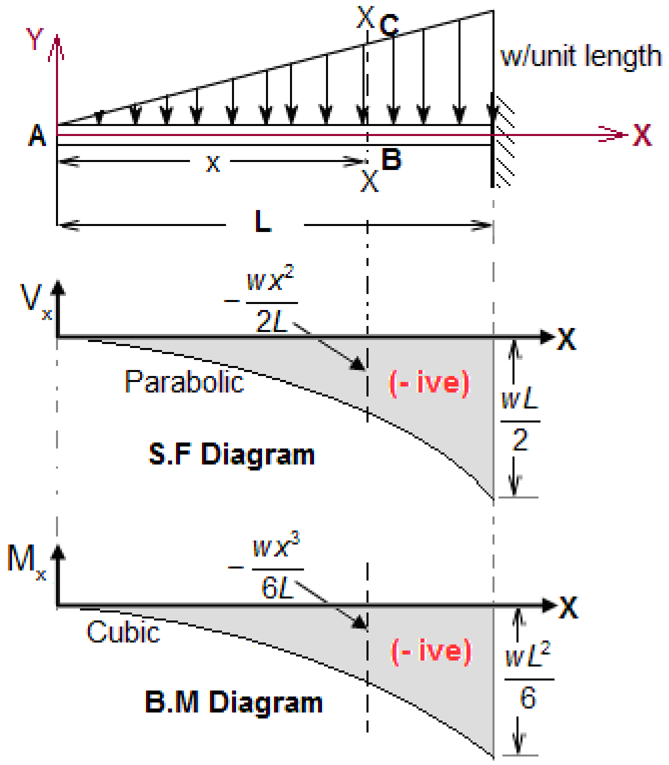
Bending moment $(M_x) =$ load × distance from centroid of triangle ABC

$= -\frac{wx^2}{2L}.\left(\frac{x}{3}\right) = -\frac{wx^3}{6L}$

∴ The bending moment variation is cubic. at

x= 0, $M_x = 0$

at x = L, $M_x = -\frac{wL^2}{6}$ i.e. Maximum Bending moment $(M_{max}) = \frac{wL^2}{6}$ at fixed end.



Alternative way: (Integration method) We know that $\frac{d(V_x)}{dx} = -\text{load} = -\frac{w}{L}.x$

or $d(V_x) = -\frac{w}{L}.x.dx$

Integrating both side

$$\int_0^{V_x} d(V_x) = -\int_0^x \frac{w}{L}.x.dx$$

or $V_x = -\frac{w}{L}.\frac{x^2}{2}$

Again we know that

$$\frac{d(M_x)}{dx} = V_x = -\frac{wx^2}{2L}$$

or $\frac{d(M_x)}{dx} = -\frac{wx^2}{2L}$

Integrating both side we get (at x=0, M_x=0)

$$\int_0^{M_x} d(M_x) = -\int_0^x \frac{wx^2}{2}.dx$$

or $M_x = -\frac{w}{2L}\times\frac{x^3}{3} = -\frac{wx^3}{6L}$

- **Load and Bending Moment diagram from Shear Force diagram**

OR

Load and Shear Force diagram from Bending Moment diagram

- If S.F. diagram consists of rectangle then the load will be point load
- If S.F diagram consists of inclined line then the load will be UDL on that portion
- If S.F diagram consists of parabolic curve then the load will be GVL

- If S.F diagram consists of cubic curve then the load distribute is parabolic.

After finding load diagram we can draw B.M diagram easily.

If B.M Diagram for a beam is given, then

4) If B.M diagram consists of inclined line then the load will be free point load

5) If B.M diagram consists of parabolic curve then the load will be U.D.L.

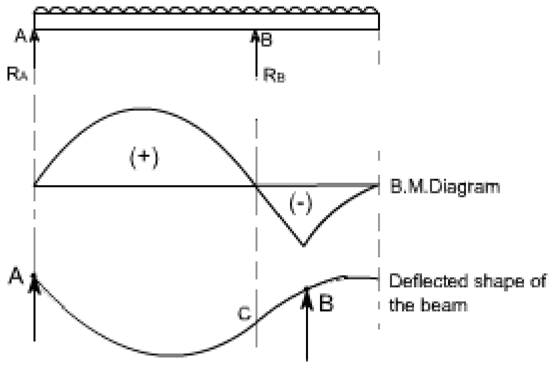
6) If B.M diagram consists of cubic curve then the load will be G.V.L.

7) If B.M diagram consists of fourth degree polynomial then the load distribution is

• Point of Contraflexure

In a beam if the bending moment changes sign at a point, the point itself having zero bending moment, the beam changes curvature at this point of zero bending moment and this point is called the point of contra flexure.

Consider a loaded beam as shown below along with the B.M diagrams and deflection diagram.



In this diagram we noticed that for the beam loaded as in this case, the bending moment diagram is partly positive and partly negative. In the deflected shape of the beam just below the bending moment diagram shows that left hand side of the beam is 'sagging' while the right hand side of the beam is 'hogging'.

The point C on the beam where the curvature changes from sagging to hogging is a point of contraflexure.

ix) There can be more than one point of contraflexure in a beam.

Chapter 5- Column

1.18 Introduction

Column. A bar or a member of a structure inclined at 90° to the horizontal and carrying an axial compressive load is called a column.

Columns are used as major elements in trusses, building frames, and sub-structure supports for bridges (e.g. piers).

Columns support compressive loads from roofs, floors, or bridge decks.

Columns transmit the vertical forces to the foundations and into the subsoil.

Strut A bar or a member of a structure inclined at other than 90° to the horizontal and carrying an axial compressive load is called a Strut.

Slenderness ratio. The ratio of the equivalent length of the column to the least radius of gyration is called the slenderness ratio.

Buckling load. The minimum axial load at which the column tends to have lateral displacement & buckle is called the buckling, crippling or critical load.

Equivalent length. It is the length of the column which gives the same , as given by a both ends hinged column.

Short Column. A column for which the slenderness ratio is less than 32 short column.

Medium Column. A column for which the slenderness ratio lies between 32 and 120 is called a medium column.

Long Column. A column for which the slenderness ratio is more than 120 called a long column.

Safe load. It is the load under which the column will not buckle.

1.19 Short and Long Columns – Modes of Failure

Column slenderness and length greatly influence a column's ability to carry load.

7 Very short, stout columns fail by crushing due to material failure.

7.10 Failure occurs once the stress exceeds the elastic (yield point) limit of the material.

8 Long, slender columns fail by buckling – a function of the column's dimensions and its modulus of elasticity.

8.10 *Buckling* is the sudden uncontrolled lateral displacement of a column atwhich point no additional load can be supported.

8.11 Failure occurs at a lower stress level than the column's material strength due to buckling (i.e. lateral instability).

Short columns

Short columns fail by crushing at very high stress levels that are above the elastic limit of the column material.

Compressive stress for short columns is based on the basic stress equation developed at the beginning of Chapter 5.

9 If the load and column size (i.e. cross-sectional area) are known, the compressive stress may be computed as

$$f_a = P_{actual}/A \quad \leq \quad F_a$$

2

where

f_a = actual compressive stress (psi or ksi) A =

cross-sectional area of the column (in^2)

P_{actual} = actual load on the column (pounds or kips)

F_a = allowable compressive stress per code (psi or ksi)

10 This stress equation can be rewritten into a design form to determine the required short column size when the load and allowable material strength are known.

$$A_{required} = P_{actual}/F_a$$

where

$A_{required}$ = minimum cross-sectional area of the column

5. 3 Long Columns – Euler Buckling

Long columns fail by buckling at stress levels that are below the elastic limit of the column material.

11 Very short column lengths require extremely large loads to cause the member to buckle.

12 Large loads result in high stresses that cause crushing rather than buckling.

Buckling in long, slender columns is due to the following.

13 Eccentricities in loading.

14 Irregularities in the column material.

Buckling can be avoided (theoretically) if the loads were applied absolutely axially, the column material was totally homogeneous with no imperfections, and construction was true and plumb.

A Swiss mathematician named Leonhard Euler (1707 – 1783) was the first to investigate the buckling behavior of slender columns within the elastic limit of the column's material.

15 Euler's equation shows the relationship between the load that causes buckling of a (pinned end) column and the material and stiffness properties of the column.

The critical buckling load can be determined by the following equation.

$$P_{critical} = \pi^2 E I_{min}/L^2$$

where

$P_{critical}$ = critical axial load that causes buckling in the column (pounds or kips) E = modulus of elasticity of the column material (psi or ksi)

I_{min} = smallest moment of inertia of the column cross-section (in^2)
(Most sections have I_x and I_y; angles have I_x, I_y and I_z.) L = column length between pinned ends (inches)

16 As the column length increases, the critical load rapidly decreases (since it is proportional to L^2), approaching zero as a limit.

17 The critical load at buckling is referred to as *Euler's critical buckling load*.

Euler's equation is valid only for long, slender columns that fail due to buckling.

18 Euler's equation contains no safety factors.

19 Euler's equation results in compressive stresses developed in columns that are well below the elastic limit of the material.

Slenderness Ratios

The ***radius of gyration*** is a geometric property of a cross section that was first introduced in Chapter 6.

$$I = Ar^2 \quad \text{and} \quad r = (I/A)^{1/2}$$

where

r = radius of gyration of the column cross section (in) I = least (minimum) moment of inertia (in^4)

A = cross-sectional area of the column (in^2)

The radius of gyration is geometric property that is used in the analysis and design of columns.

Using the radius of gyration, the critical stress developed in a long column at buckling can be expressed by the following equation.

$$f_{critical} = P_{critical}/A = \pi^2 E I_{min}/AL^2 = \pi^2 E(Ar^2)/AL^2 = \pi^2 E/(L/r)^2$$

The term "L/r" is known as the *slenderness ratio*.

20 A <u>higher</u> slenderness ratio means a <u>lower</u> critical stress that will cause buckling.

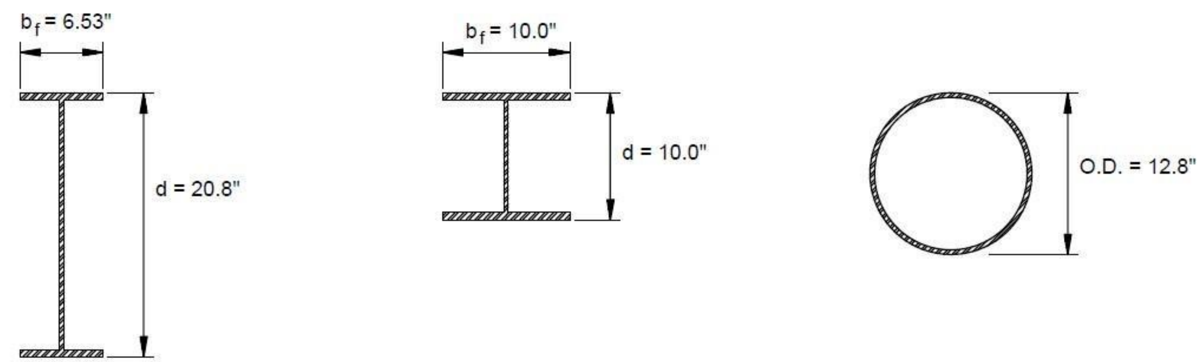
21 Conversely, a <u>lower</u> slenderness ratio results in a <u>higher</u> critical stress (but still within the elastic range of the material).

Column sections with large r-values are more resistant to buckling.

22 Compare the difference in r_{min} values and slenderness ratios for the three column cross sections shown below.

22.10 All three cross sections have relatively equal cross-sectional areas but very different radii of gyration about the critical buckling axis.

22.11 All three columns are assumed to be 15 feet in length and pin-connected at both ends.



W 21 x 50	W 10 x 49	12" Dia. Std. Pipe
$A = 14.7\ in^2$	$A = 14.4\ in^2$	$A = 13.6\ in^2$
$r_x = 8.18"$	$r_x = 4.35"$	$r_x = 4.39"$
$r_y = 1.30"$	$r_y = 2.54"$	$r_y = 4.39"$
$\frac{L}{r} = \frac{15\ (12"/')}{1.30} = 138.5$	$\frac{L}{r} = \frac{15\ (12"/')}{2.54} = 70.9$	$\frac{L}{r} = \frac{15\ (12"/')}{4.39} = 41.1$

Comparison of steel cross sections with equivalent areas

The most efficient column sections for axial loads are those with almost equal r_x and r_y values.

23 Circular pipe sections and square tubes are the most effective shapes since the radii of gyration about both axes are the same ($r_x = r_y$).

24 Circular pipe sections and square tubes are often used as columns for light to moderate loads.

Wide-flange shapes may be preferred despite the structural advantages of closed cross-sectional shapes (like tubes and pipes).

25 The practical considerations of wide-flange shapes include the following.

25.10 Wide-flange sections support heavy loads.

25.11 Wide-flange sections accommodate beam connections.

Special wide-flange sections are specifically manufactured to provide relatively symmetrical columns (r_x/r_y ratios approaching 1.0) with large load-carrying capability.

26 Most of these column sections (generally W8, W10, W12, and W14) have depth and flange widths approximately equal (i.e. a "boxy" configuration).

- **End Support Conditions and Lateral Bracing**

Previously, each column was assumed to have pinned ends in which the member ends were free to rotate (but not translate) in any direction at their ends.

- When the column buckles, it will do so in one smooth curve.
- The length of this curve is referred to as the *effective length*.

In practice, a column may not be pinned at the ends.

- The column length free to buckle is greatly influenced by its end support conditions.
- The load-carrying capacity of a column is affected by the end support conditions.
  - Restraining the ends of a column with a fixed support increases the load- carrying capacity of a column.
  - Allowing translation as well as rotation (i.e. free end) at one end of a column generally reduces its load-carrying capacity.

Column design formulas generally assume a condition in which both ends are pinned.

- When other conditions exist, the load-carrying capacity is increased or decreased and the allowable compressive stress is increased or decreased.
- A factor K is used as a multiplier for converting the actual column length to an effective buckling length based on end conditions.

The American Institute of Steel Construction (AISC) provides recommended effective length factors when ideal conditions are approximated.

- The six cases are presented as follows.

Case A: Both ends are pinned.

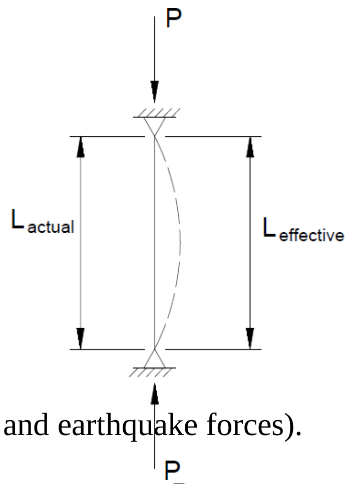
The structure is adequately braced against lateral forces (e.g. wind and earthquake forces).

Theoretical K-value: $K = 1.0$

Effective length: $L_e = L$

$P_{critical} = \pi^2 EI_{min}/L^2$

Examples: Timber column nailed top and bottom; steel column with simple clip angle connection top and bottom.



Case B: Both ends are fixed.

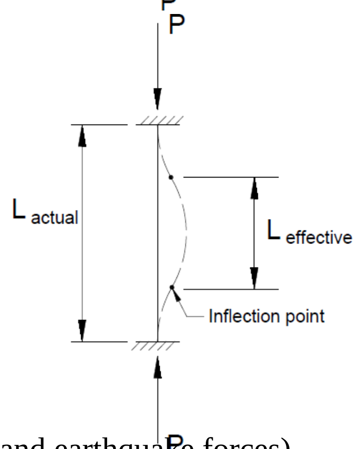
The structure is adequately braced against lateral forces (e.g. wind and earthquake forces).

Theoretical K-value: K= 0.5 Effective

length: $L_e = 0.5\ L$ $P_{critical}$

$= \pi^2 EI_{min}/(0.5L)^2 = 4\pi^2 EI_{min}/L^2$

Examples: Concrete column rigidly (monolithically cast) connected to large beams top and bottom; steel column rigidly connected (welded) to large steel beams top and bottom.



Case C: One end is pinned and one end is fixed.

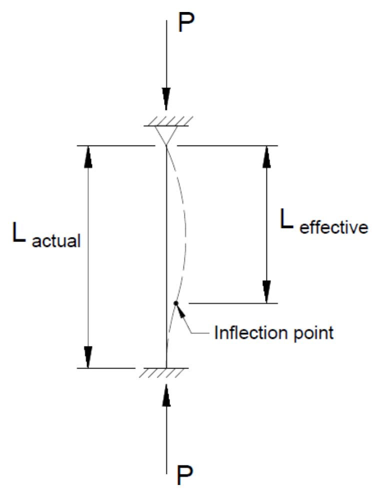
The structure is adequately braced against lateral forces (e.g. wind and earthquake forces).

Theoretical K-value: K = 0.7 Effective

length: $L_e = 0.707\ L$ $P_{critical}$

$= \pi^2 EI_{min}/(0.707L)^2 = 2\pi^2 EI_{min}/L^2$

Examples: Concrete column rigidly connected to concrete slab at the base and attached to light- gauge roofing at the top.



Case D: One end is free and one end is fixed.
Lateral translation is possible.

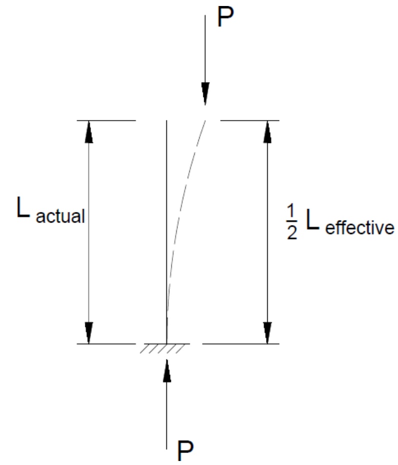
- An eccentric column load is developed.

Theoretical K-value: $K = 2.0$ Effective

length: $L_e = 2.0\ L$ $P_{critical}$

$= \pi^2 EI_{min}/(2L)^2 = \pi^2 EI_{min}/4L^2$

Examples: Water tank mounted on a simple pipe column; flagpole.



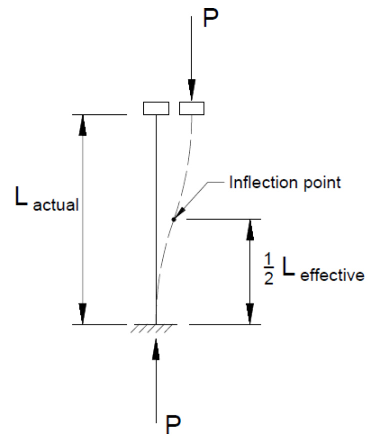
Case E: Both ends are fixed with some lateral translation.

Theoretical K-value: $K = 1.0$

Effective length: $L_e = 1.0\ L$

$P_{critical} = \pi^2 EI_{min}/L^2$

Examples: Flexible column attached to a rigid beam and supported by a fixed base.



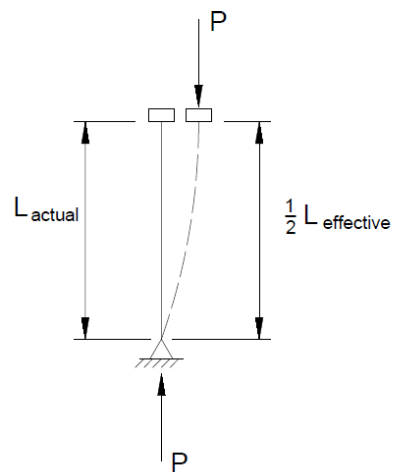
Case F: The base is pinned and the top is fixed with some lateral translation.

Theoretical K-value: $K = 2.0$

Effective length: $L_e = 2.0\ L$

$P_{critical} = \pi^2 EI_{min}/(2L)^2 = \pi^2 EI_{min}/4L^2$

Examples: Steel column with a rigid connection to a beam above and a simple pin connection at the base.



Chapter – 6 Bending Stress in Beam

- **Euler Bernoulli's Equation or (Bending stress formula) or Bending Equation**

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

Where σ = Bending Stress

M = Bending Moment

I = Moment of Inertia

E = Modulus of elasticity

R = Radius of curvature

y = Distance of the fibre from NA (Neutral axis)

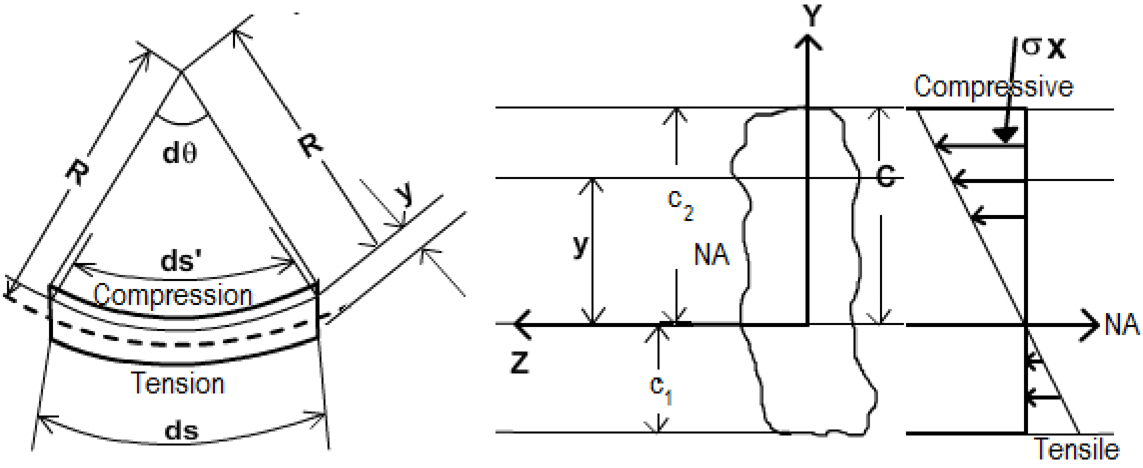
- **Assumptions in Simple Bending Theory**

All of the foregoing theory has been developed for the case of pure bending i.e constant B.M along the length of the beam. In such case

- The shear force at each c/s is zero.
- Normal stress due to bending is only produced.
- Beams are initially straight
- The material is homogenous and isotropic i.e. it has a uniform composition and its mechanical properties are the same in all directions
- The stress-strain relationship is linear and elastic
- Young's Modulus is the same in tension as in compression

- Sections are symmetrical about the plane of bending
- Sections which are plane before bending remain plane after bending

•



$$o_{\max} = \sigma_t = \frac{Mc_1}{I}$$

$$o_{\min} = \sigma_c = \frac{Mc_2}{I}$$ (Minimum in sense of sign)

- **Section Modulus (Z)**

$$Z = \frac{I}{y}$$

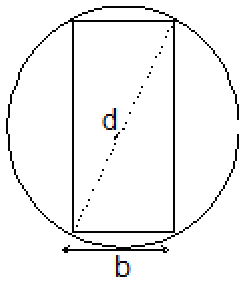
- Z is a function of beam c/s only
- Z is other name of the strength of the beam
- The strength of the beam sections depends mainly on the section modulus

- The flexural formula may be written as, $o = \frac{M}{Z}$

- Rectangular c/s of width is "b" & depth "h" with sides horizontal, Z = $\frac{bh^2}{6}$

- Square beam with sides horizontal, Z = $\frac{a^3}{6}$
- Square c/s with diagonal horizontal, Z = $\frac{a^3\sqrt{2}}{6}$
- Circular c/s of diameter "d", Z = $\frac{\pi d^3}{32}$

A log diameter "d" is available. It is proposed to cut out a strongest beam from it. Then



$$Z = \frac{b(d^2 - b^2)}{6}$$

Therefore, $Z_{max} = \frac{bd^3}{9}$ for $b = \frac{d}{\sqrt{3}}$

- **Flexural Rigidity (EI)**

Reflects both

- Stiffness of the material (measured by E)
- Proportions of the c/s area (measured by I)

- **Axial Rigidity = EA**

- **Beam of uniform strength**

It is one is which the maximum bending stress is same in every section along the longitudinal axis.

For it $M \alpha \text{bh}^2$

Where b = Width of beam

h = Height of beam

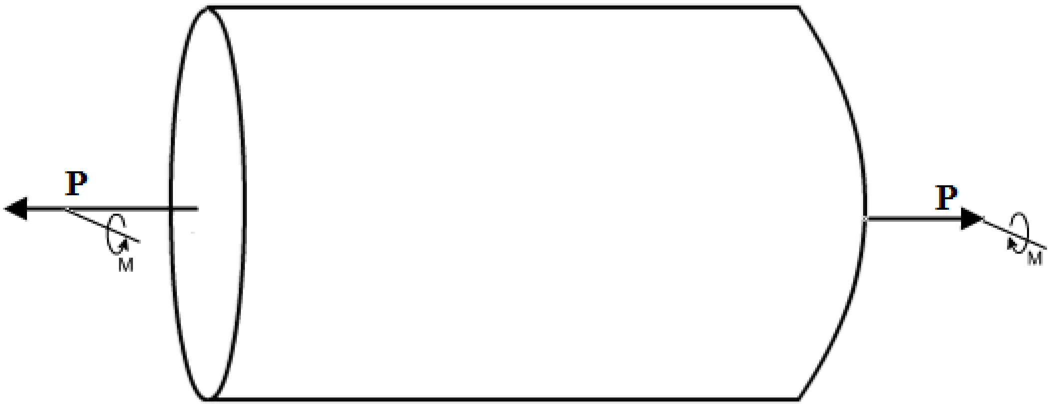
To make Beam of uniform strength the section of the beam may be varied by

- Keeping the width constant throughout the length and varying the depth, *(Most widely used)*
- Keeping the depth constant throughout the length and varying the width
- By varying both width and depth suitably.

• Bending stress due to additional Axial thrust (P).

A shaft may be subjected to a combined bending and axial thrust. This type of situation arises in various machine elements.

If P = Axial thrust



Then direct stress (σ_d) = P / A (stress due to axial thrust)

This direct stress (σ_d) may be tensile or compressive depending upon the load P is tensile or compressive.

And the bending stress $(\sigma_b) = \frac{My}{I}$ is varying linearly from zero at centre and extremum (minimum or maximum) at top and bottom fibres.

If P is compressive then

- At top fibre ♦ $= \frac{P}{A} + \frac{My}{I}$ (compressive)
- At mid fibre • $= \frac{P}{A}$ (compressive)
- At bottom fibre $\sigma = \frac{P}{A} - \frac{My}{I}$ (compressive)

• Load acting eccentrically to one axis

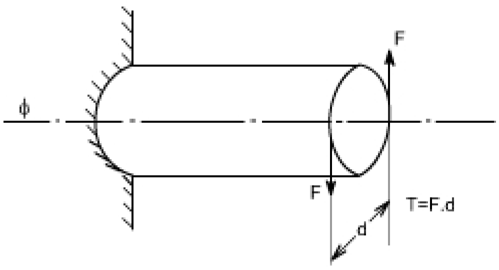
- $\sigma_{max} = \frac{P}{A} + \frac{(P \times e)\, y}{I}$ Where 'e' is the eccentricity at which 'P' is act.
- $\sigma_{min} = \frac{P}{A} - \frac{(P \times e)\, y}{I}$

Chapter- 7

Torsion

- **Torsion of circular shafts**

Definition of Torsion: Consider a shaft rigidly clamped at one end and twisted at the other end by a torque T = F.d applied in a plane perpendicular to the axis of the bar such a shaft is said to be in torsion.



Effects of Torsion: The effects of a torsional load applied to a bar are

9. To impart an angular displacement of one end cross – section with respect to the other end.

10. To setup shear stresses on any cross section of the bar

perpendicular to its axis.

(ii) <u>GENERATION OF SHEAR STRESSES</u>

The physical understanding of the phenomena of setting up of shear stresses in a shaft subjected to a torsion may be understood from the figure 1-3.

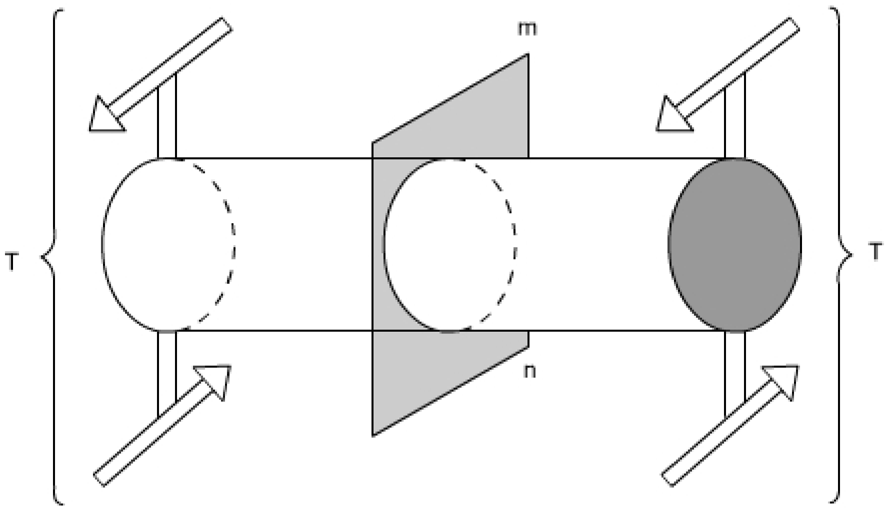


Fig 1: Here the cylindrical member or a shaft is in static equilibrium where T is the resultant external torque acting on the member. Let the member be imagined to be cut by some imaginary plane 'mn'.

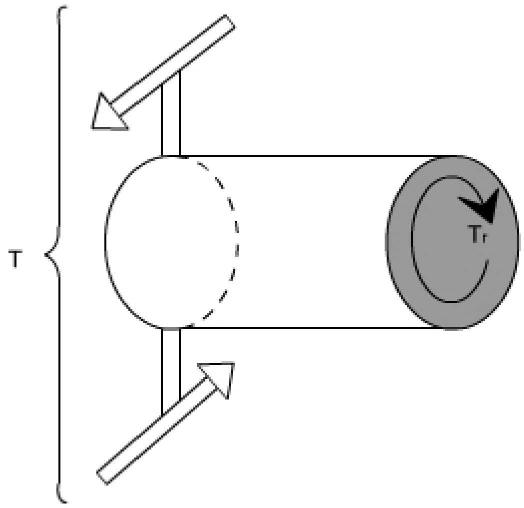


Fig 2: When the plane ‘mn' cuts remove the portion on R.H.S. and we get a fig 2. Now since the entire member is in equilibrium, therefore, each portion must be in equilibrium. Thus, the member is in equilibrium under the action of resultant external torque T and developed resisting Torque T_r .

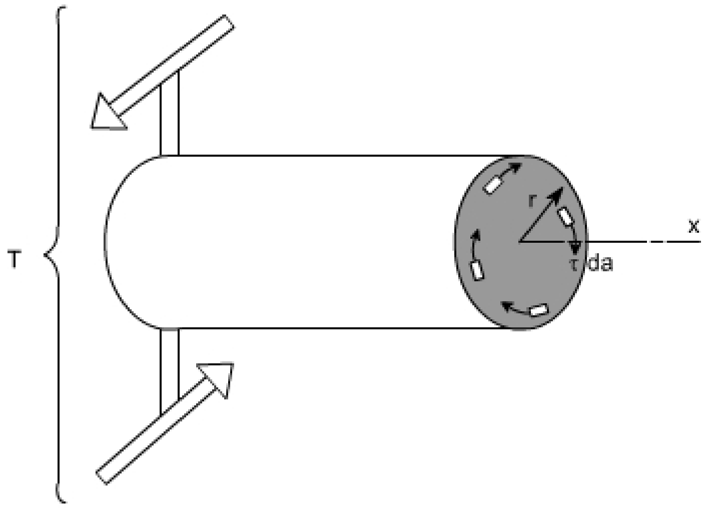


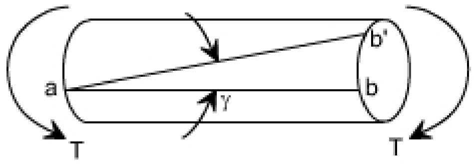
Fig 3: The Figure shows that how the resisting torque T_r is developed. The resisting torque T_r is produced by virtue of an infinites mal shear forces acting on the plane perpendicular to the axis of the shaft. Obviously such shear forces would be developed by virtue of sheer stresses.

Therefore we can say that when a particular member (say shaft in this case) is subjected to a torque, the result would be that on any element there will be shear stresses acting. While on other faces the complementary sheer forces come into picture. Thus, we can say that when a member is subjected to torque, an element of this member will be subjected to a state of pure shear.

Shaft: The shafts are the machine elements which are used to transmit power in machines.

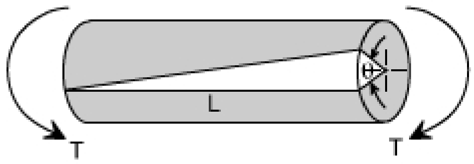
Twisting Moment: The twisting moment for any section along the bar / shaft is defined to be the algebraic sum of the moments of the applied couples that lie to one side of the section under consideration. The choice of the side in any case is of course arbitrary.

Shearing Strain: If a generator a – b is marked on the surface of the unloaded bar, then after the twisting moment 'T' has been applied this line moves to ab'. The angle ‘□measured in radians, between the final and original positions of the generators is defined as the shearing strain at the surface of the bar or shaft. The same definition will hold at any interior point of the bar.



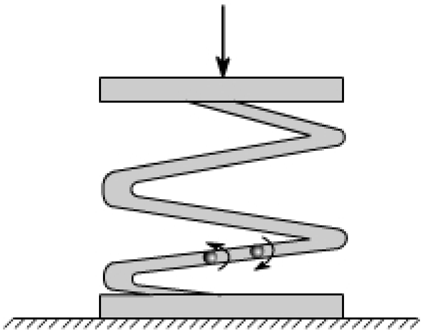
- **Modulus of Elasticity in shear:** The ratio of the shear stress to the shear strain is called the modulus of elasticity in shear OR Modulus of Rigidity and in represented by the symbol $G = \frac{\tau}{r}$

Angle of Twist: If a shaft of length L is subjected to a constant twisting moment T along its length, than the angle through which one end of the bar will twist relative to the other is known is the angle of twist.



- Despite the differences in the forms of loading, we see that there are number of similarities between bending and torsion, including for example, a linear variation of stresses and strain with position.
  In torsion the members are subjected to moments (couples) in planes normal to their axes.
- For the purpose of desiging a circular shaft to withstand a given torque, we must develop an equation giving the relation between twisting moment, maximum shear stress produced, and a quantity representing the size and shape of the cross-sectional area of the shaft.

Not all torsion problems, involve rotating machinery, however, for example some types of vehicle suspension system employ torsional springs. Indeed, even coil springs are really curved members in torsion as shown in figure.



- Many torque carrying engineering members are cylindrical in shape. Examples are drive shafts, bolts and screw drivers

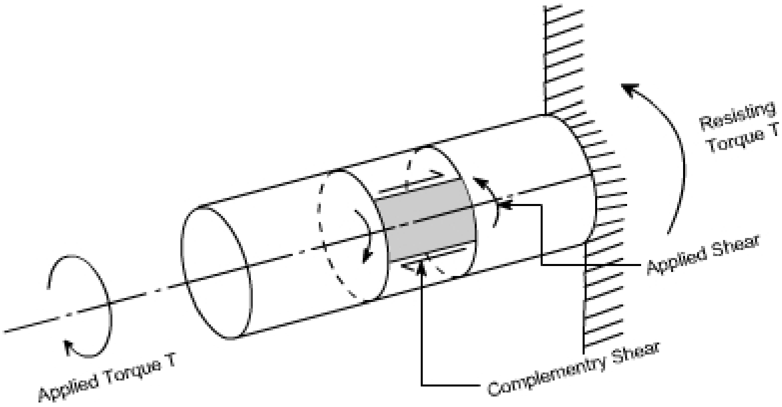
  - **Simple Torsion Theory or Development of Torsion Formula :** Here we are basically interested to derive an equation between the relevant parameters

- **Relationship in Torsion:** $\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$

1 st Term: It refers to applied loading ad a property of section, which in the instance is the polar second moment of area. **2 nd Term:** This refers to stress, and the stress increases as the distance

from the axis increases.

3 rd Term: it refers to the deformation and contains the terms modulus of rigidity & combined term (□ □ l) which is equivalent to strain for the purpose of designing a circular shaft to with stand a given torque we must develop an equation giving the relation between Twisting moments max m shear stain produced and a quantity representing the size and shape of the cross – sectional area of the shaft.



Refer to the figure shown above where a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear, the moment of resistance developed by the shear stresses being every where equal to the magnitude, and opposite in sense, to the applied torque. For the purpose of deriving a simple theory to describe the behavior of shafts subjected to torque it is necessary make the following base assumptions.

Assumption:

2 The materiel is homogenous i.e of uniform elastic properties exists throughout the material.

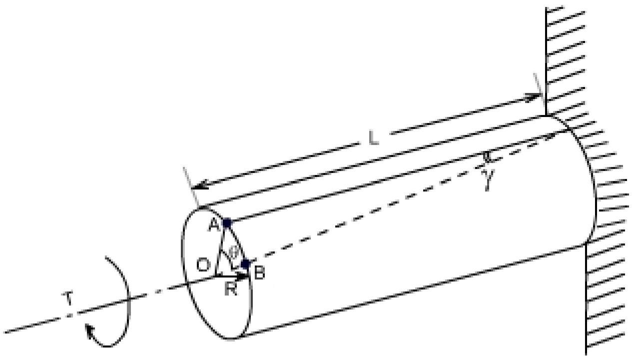
3 The material is elastic, follows Hook's law, with shear stress proportional to shear strain.

4 The stress does not exceed the elastic limit.

5 The circular section remains circular

6 Cross section remain plane.

7 Cross section rotate as if rigid i.e. every diameter rotates through the same angle.



Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed Under the action of this torque a radial line at the free end of the shaft twists through an angle □ , point A moves to B, and AB subtends an angle ‘ □ ' at the fixed end. This is then the angle of distortion of the shaft i.e the shear strain.

Since angle in radius = arc / Radius

arc AB = R□

= L □ [since L and □ also constitute

the arc AB] Thus, □ = R□ / L(1)

From the definition of Modulus of rigidity or Modulus of elasticity in shear

$$G = \frac{\text{shear stress}(\tau)}{\text{shear strain}(\gamma)}$$

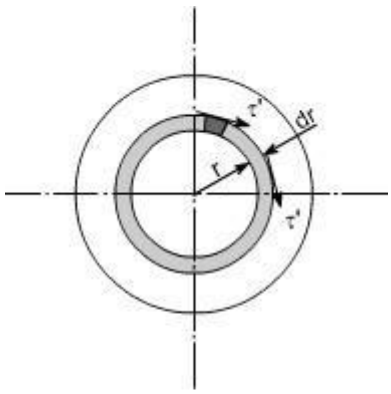
where γ is the shear stress set up at radius R.

Then $\frac{\tau}{G} = \gamma$

Equating the equations (1) and (2) we get $\frac{R\theta}{L} = \frac{\tau}{G}$

$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right)$ where τ' is the shear stress at any radius r.

Stresses: Let us consider a small strip of radius r and thickness dr which is subjected to shear stress □'.



The force set up on each element

= stress x area

The total torque T on the section, will be the sum of all the c $T = \int_0^R 2\pi\tau' r^2 dr.$

Since □' is a function of r, because it varies with radius so writing down □' in terms of r from the equation (1).

$$\text{i.e } \tau' = \frac{G\theta . r}{L}$$

$$\text{we get } T = \int_0^R 2\pi \frac{G\theta}{L} . r^3 \, dr$$

$$T = \frac{2\pi G\theta}{L} \int_0^R r^3 \, dr$$

$$= \frac{2\pi G\theta}{L} . \left[\frac{R^4}{4}\right]_0^R$$

$$= \frac{G\theta}{L} . \frac{2\pi R^4}{4}$$

$$= \frac{G\theta}{L} . \frac{\pi R^4}{2}$$

$$= \frac{G\theta}{L} . \left[\frac{\pi d^4}{32}\right] \text{ now substituting } R = d/2$$

$$= \frac{G\theta}{L} . J$$

$$\text{since } \frac{\pi d^4}{32} = J \text{ the polar moment of inertia}$$

$$\text{or } \frac{T}{J} = \frac{G\theta}{L} \qquad \text{......(2)}$$

if we combine the equation no.(1) and (2) we get $\boxed{\frac{T}{J} = \frac{\tau'}{r} = \frac{G.\theta}{L}}$

ere
T = applied external Torque, which is constant over Length L;

J = Polar moment of Inertia $= \frac{\pi d^4}{32}$ for solid shaft

$= \frac{\pi (D^4 - d^4)}{32}$ for a hollow shaft.

[D = Outside diameter ; d = inside diameter] G = Modules of rigidity (or Modulus of elasticity shear)

θ = It is the angle of twist in radians on a length L.

Tensional Stiffness: The tensional stiffness k is defined as the torque

per radius twist i.e, $k = T / \theta = GJ / L$

Power Transmitted by a shaft : If T is the applied Torque and ω is the angular velocity of the shaft, then the power transmitted by the shaft is

$$P = T . \omega = \frac{2\pi NT}{60} = \frac{2\pi NT}{60.10^3} \text{ kw}$$

where N = rpm

Chapter- 8

Springs

Spring Definition:

Spring can be defined as an elastic member, whose main function is to **deflect under the action of load** and **recovers it's original shape when the load is removed**. It is a mechanical component.

The function of Spring:

Spring can be used for:

- To absorb the shocks or viberation as in-car springs, railway buffers, etc.
- To measure the forces as in a spring balance.
- Apply forces in brakes and clutches to stop the vehicles.
- The function is to store the energy as in clocks, toys, etc.
- Control the motion as in cams and followers by maintaining contact between two elements.

Different types of Spring:

Spring can be classified into these following types*:*

- Helical Spring
- Closed Coil Helical Spring
- Open Coil Helical Spring
- Torsion Spring
- Laminated or leaf Spring
- Disc or Belleville Spring
- Special Purpose Spring

Helical Spring:

A helical spring is a mechanical device which is typically used to store energy and subsequently release it, to absorb shock, or to maintain a force between contacting surfaces.

Closed Coil Helical Spring

In this spring coil is wound tightly providing no visible gap between two adjacent coils, then the same is called **close coiled spring** or extension spring. These spring are designed to carry tensile load . Helix angle of this spring is usually 10° or below that. This type of spring is commonly used in heavy duty applications such as:

- Garage doors
- Vice-grip pliers
- Self-closing door hinges
- Bike or cycle stand spring

Open Coil Helical Spring

when sufficient gap is provided between two adjacent coils, then the same is called **open coiled spring** or compression spring. Therefore, both are helical springs, but with varying space between two adjacent coils. Helix angle of this spring is more than 10°. Open coiled springs are designed to undergo extension and compression; accordingly under the action of axial load.

This type of spring is commonly used in low duty applications such as:

- Spring-operated ball point pen
- Bike shock absorber
- Certain cam-followers, valves, brakes and clutches

Torsion Springs:

The characteristics of such springs are that they tend to wind up by the load. They can be either helical or spiral in shape. These types of springs are used in circuit breaker mechanisms.

Leaf springs:

These types of springs are comprised of metal plates of different lengths held together with the help of bolts and clamps. Commonly seen being used as suspensions for vehicles.

Disc Springs:

As the name suggests such types of springs are comprised of conical discs held together by a bolt or tube.

Special Purpose Springs:

These springs are altogether made of different materials such as air and water.

Spring Realted Terms

Stiffness (S) It is the ration of Load per unit deflection

$$S= W/\delta$$

Mean Coil Diameter:- it is the total diameter of the spring. It is denoted by D.

Wire Coil Diameter:- It is total diameter of a wire. It is denoted by d.

Spring Index:- It is the ration of Mean coil diameter to the Wire coil diameter.

$$K=D/d$$

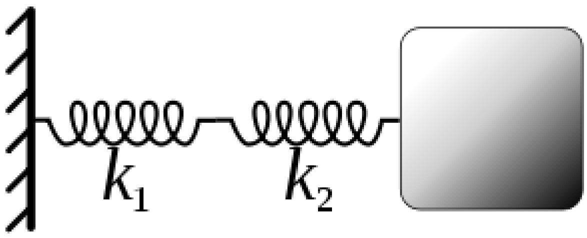
Solid Length of Spring:- it is the length of spring when there will no gap between spring coils.

Soild Length of spring = $2\pi Rn$

Where n is no of coils of spring.

Free Length of Spring – it is the length of spring when the spring has gap between number of coils

Spring in Series:- If two springs of stiffness K1 and K2 are connected in series is subjected to load W , spring are said to be in series combination.



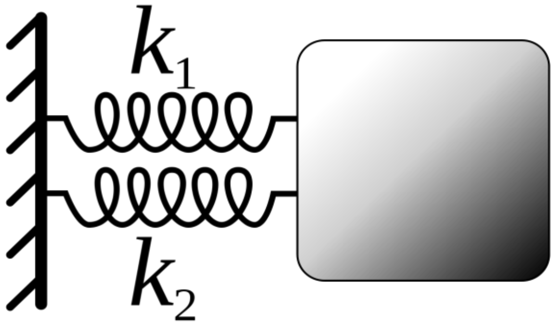
Total Deflection of Spring $\delta = \delta 1 + \delta 2$

$$W/K = W/K1 + W/K2$$

Where K is the total stiffness of the spring

$$1/K = 1/K1 + 1/K2$$

Spring in Parallel:- When two springof Stiffness K1 and K2 are connected in parallel mode as shown in fig and subjected to al load W is known parallel combination of spring . In these combination spring are deflected equally.



Total load (W) shared by each spring is = W1+W2

Total Stiffness of Spring is (K)=   K1 + K2

Spring Materials:

Springs are made of the following materials:

- **Hard drawn wire** (These are general purpose spring, where we need low-temperature resistance and low stress we can use these types of spring material)
- **Stainless steel**

- **Phosphor Bronze** (These types of spring are used in Electrical industry due to the material have good electrical conductivity and good resistance to corrosion)
- **Chrome Silicon** (This type of spring can be used in high temperature up to 250-degree celsius)
- **Chrome Vanadium** (These types of spring have good thermal conductivity up to 220-degree Celsius and also has good fatigue resistance)

Applications of Spring:

Spring can be used in various field such as:

- Automotives
- Computer keyboards
- Mattresses
- Appliances (toasters)
- Door handles
- Compressors
- Pens
- Clock
- Vehicle suspension
- Watches
- Mini Drill
- Lock mechanisms: Key-recognition and for coordinating the movements of various parts of the lock.
- Pop-open devices: CD players, Tape recorders etc.
- Spring mattresses
- Slinky
- Trampoline
- Buckling spring keyboards
- Upholstery: Upholstery coil springs
- Echo effect in electronic organs
- Spring can be used for storing of energies

- Spring can use to measure forces in the weighing machine**.**

Advantages of Spring:

These are the following advantages of spring:

- Produce a good cushioning effect
- Spring has good shock-absorbing ability
- Latching
- Locking
- Holding
- Easy to design
- Cheaper to Produce